STABILIZATIONS VIA LEFSCHETZ FIBRATIONS AND EXACT OPEN BOOKS

SELMAN AKBULUT AND M. FIRAT ARIKAN

Abstract. We show that if a contact open book (Σ, h) on a $(2n+1)$-manifold M $(n \geq 1)$ is induced by a Lefschetz fibration $\pi : W \to D^2$, then there is a one-to-one correspondence between positive stabilizations of (Σ, h) and *positive stabilizations* of π. More precisely, any positive stabilization of (Σ, h) is induced by the corresponding positive stabilization of π, and conversely any positive stabilization of π induces the corresponding positive stabilization of (Σ, h). We define *exact open books* as boundary open books of exact Lefschetz fibrations, and show that any exact open book carries a contact structure. Moreover, we prove that there is a one-to-one correspondence (similar to the one above) between *convex stabilizations* of an exact open book and *convex stabilizations* of the corresponding exact Lefschetz fibration. We also show that convex stabilization of exact Lefschetz fibrations produces symplectomorphic manifolds.

1. Introduction

In the last decade the correspondence given by Giroux [11], between contact structures and open book decompositions have led to many developments in understanding the relations between the contact geometry and the topology of the underlying odd dimensional closed manifolds. This correspondence is much stronger in dimension three and has been used as a bridge between four dimensional geometries and topology, leading much progress in understanding of different types of fillability and Lefschetz type fibrations.

One of the main features used in the above correspondence is positive stabilization. Namely, if we positively stabilize an open book (Σ, h) carrying a contact structure ξ on a closed 3-manifold M, then the resulting open book still carries ξ. Such stabilizations can be interpreted as taking the contact connect sum of (M, ξ) with (S^3, ξ_{st}) where ξ_{st} is the unique tight (Stein fillable) contact structure on the 3-sphere S^3. In terms of open books, this corresponds to taking the Murasugi sum (or plumbing) of (Σ, h) with the open book (H^+, τ_C) on S^3 where H^+ is the positive (left-handed) Hopf band and τ_C denotes the right-handed Dehn twist along the core circle C in H^+.

To get analogous statements for higher dimensions, one can replace (H^+, τ_C) with its generalization $\mathcal{OB}$, which is an open book carrying the standard contact structure ξ_0 on $(2n+1)$-sphere S^{2n+1} and obtained from a certain Milnor fibration. The pages of $\mathcal{OB}$ are diffeomorphic to the closed tangent unit disk bundle $\mathcal{D}(TS^n)$ over S^n and its monodromy is the (generalized) right-handed Dehn twist along the zero section (see below). Then

Date: December 22, 2011.
1991 *Mathematics Subject Classification.* 58D27, 58A05, 57R65.
Key words and phrases. Contact & symplectic structures, open book, Lefschetz fibration, stabilization.
The first author is partially supported by NSF FRG grant DMS- 0905917.
The second author is partially supported by NSF FRG grant DMS-1065910.

one can define a positive stabilization of an open book (Σ, h) carrying a contact structure ξ on an $(2n+1)$-dimensional closed manifold M^{2n+1} by taking the Murasugi sum of (Σ, h) with $\mathcal{OB}$, along a properly embedded Lagrangian n-ball L in Σ with Legendrian boundary and a fiber in $\mathcal{D}(TS^n)$. Again this amounts to taking the contact connect sum of (M^{2n+1}, ξ) with (S^{2n+1}, ξ_0) and stabilized open book still carries ξ [11]. In terms of contact surgery and Weinstein handles, a positive stabilization corresponds to performing (resp. attaching) a pair of subcritical and critical surgeries (resp. Weinstein handles) which cancels each other (see [18] for a proof).

One of the missing part of this picture is the relation of such operations to Lefschetz fibrations. The aim of the present work is to provide some results to fill this gap. Through out the paper, the base space of any Lefschetz fibration is assumed to be the 2-disk, and we focus only on the open books which are induced by Lefschetz fibrations. We study the open book $\mathcal{OB}$ (which is induced by a certain Lefschetz fibration $\mathcal{LF}$ on the standard $(2n+2)$-ball) in Section 2 where we also recall positive stabilizations of open books and the characterization of Lefschetz fibrations. In Section 3, we explicitly define a process, called *positive stabilization* on Lefschetz fibrations and show that there is a one-to-one correspondence between positive stabilizations of open books and Lefschetz fibrations. *Exact open books* are introduced in Section 4 as boundaries of exact Lefschetz fibrations. After recalling Weinstein handles and isotropic setups briefly in Section 5, we will get a similar correspondence for exact open books and exact Lefschetz fibrations in Section 6, where we also define *convex stabilization* as an exact symplectic version of positive stabilization. We remark that any observation we will make here is also true for dimensions 3 and 4, and so the work done here can be thought as canonical generalizations of the corresponding 3- and 4-dimensional results to higher dimensions.

2. Preliminaries

2.1. **The open book $\mathcal{OB}$ and the associated Lefschetz fibration $\mathcal{LF}$.** Consider the polynomial P on the complex space $\mathbb{C}^{n+1}$ (for $n \geq 1$) given by

$$P(z_1, ..., z_{n+1}) = z_1^2 + z_2^2 + \cdots + z_{n+1}^2.$$

It is clear that the only critical point of P occurs at the origin, and so the intersection of the zero set $Z(P)$ of P with the sphere

$$\mathbb{S}^{2n+1}_{\varepsilon} = \{|z_1|^2 + |z_2|^2 + \cdots + |z_{n+1}|^2 = \varepsilon^2\},$$

where $\varepsilon > 0$ is small enough, is a smooth manifold K of dimension $2n-1$. K is a member of a family known as Brieskorn manifolds introduced in [3]. It is known by [15] that the complement $\mathbb{S}^{2n+1}_{\varepsilon} \setminus K$ of K fibers over the unit circle $S^1 \subset \mathbb{C}$ via the map $\Theta : \mathbb{S}^{2n+1}_{\varepsilon} \setminus K \to S^1$ given by

$$\Theta(z_1, ..., z_{n+1}) = \frac{P(z_1, ..., z_{n+1})}{|P(z_1, ..., z_{n+1})|}.$$

Let $\mathcal{OB}$ be the open book on $\mathbb{S}^{2n+1}_{\varepsilon}$ determined by the pair (K, Θ). For any $e^{i\theta} \in S^1$, the Milnor fiber (or the *page* of $\mathcal{OB}$) $F_\theta := \Theta^{-1}(e^{i\theta})$ is parallelizable and has the homotopy type of S^n [15], and indeed it can be identified with the total space of the tangent bundle TS^n of the n-sphere S^n (e.g. [9], p. 81). By considering the closure $\bar{F}_\theta$ as the closed tangent unit disk bundle $\mathcal{D}(TS^n)$ over S^n, we can identify the *binding* K of $\mathcal{OB}$ as the tangent unit sphere bundle $\mathcal{S}(TS^n)$ over S^n. For our purposes we identify TS^n with the cotangent bundle T^*S^n by using the natural duality, and assume that each page $\bar{F}_\theta$ of $\mathcal{OB}$ is diffeomorphic to the cotangent unit disk bundle $\mathcal{D}(T^*S^n)$, and so the binding K is diffeomorphic to the cotangent unit sphere bundle $\mathcal{S}(T^*S^n)$.

Now we define the function $\Pi : \mathbb{B}_\varepsilon^{2n+2} \to D^2$ from $(2n+2)$-ball

$$\mathbb{B}_\varepsilon^{2n+2} = \{|z_1|^2 + |z_2|^2 + \cdots + |z_{n+1}|^2 \leq \varepsilon^2\} \subset \mathbb{C}^{n+1}$$

onto the unit disk $D^2 \subset \mathbb{C}$ by restricting P and then normalizing by ε^2, that is,

$$\Pi = \frac{1}{\varepsilon^2} P|_{\mathbb{B}_\varepsilon^{2n+2}}.$$

By definition (see [12], for instance) Π is a local model for a Lefschetz fibration over the unit disk having only one singular fiber over the origin. Also *regular* fibers $\Pi^{-1}(z), z \neq 0$, are diffeomorphic to $\mathcal{D}(T^*S^n)$ because Π is a topological locally trivial fibration on $D^2 \setminus \{0\}$ [13] (e.g. [9], Chapter 3). Therefore, Π defines a Lefschetz fibration $\mathcal{LF}$ on $\mathbb{B}_\varepsilon^{2n+2}$ which induces the open book $\mathcal{OB}$ on the boundary sphere $\mathbb{S}_\varepsilon^{2n+1}$. By definition, the monodromy of $\mathcal{LF}$ is the monodromy of $\mathcal{OB}$. According to [8, 12] this monodromy is (up to isotopy) equal to the right-handed Dehn twist

$$\delta : \mathcal{D}(T^*S^n) \to \mathcal{D}(T^*S^n)$$

along the *vanishing cycle* which is the zero section (a copy of S^n) in $\mathcal{D}(T^*S^n)$. To describe δ precisely, identify the interior of a page with T^*S^n and write the points in T^*S^n as $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ such that $|\mathbf{q}| = 1$ and $\mathbf{q} \perp \mathbf{p}$. Then

$$\delta(\mathbf{q}, \mathbf{p}) = \begin{pmatrix} \cos g(|\mathbf{p}|) & |\mathbf{p}|^{-1} \sin g(|\mathbf{p}|) \\ -|\mathbf{p}| \sin g(|\mathbf{p}|) & \cos g(|\mathbf{p}|) \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$$

where g is a smooth function that increases monotonically from π to 2π on some interval, and outside this interval g is identically equal to π or 2π. Observe that δ is the antipodal map on the zero section $S^n \times \{\mathbf{0}\} = \{(\mathbf{q}, \mathbf{p}) \,|\, |\mathbf{q}| = 1, \mathbf{p} = \mathbf{0}\}$, while it is the identity map for $|\mathbf{p}|$ large. Note that as abstract open book $\mathcal{OB}$ is determined by the pair $(\mathcal{D}(T^*S^n), \delta)$.

Now let $z_j = x_j + iy_j$ for $j = 1, ..., n+1$. Then with respect to the complex coordinates $z = (z_1, ..., z_{n+1})$, the *standard Stein structure* on $\mathbb{C}^{n+1}$ (and hence on $\mathbb{B}_\varepsilon^{2n+2}$) is defined by the pair

$$(J_0, \psi_0) = (i \times \cdots \times i, |\, z|^2).$$

This defines the *standard symplectic* (indeed *Kähler*) *form*

$$\omega_0 = -d(d\psi_0 \circ J_0) = \sum_{j=1}^{n+1} dx_j \wedge dy_j$$

whose *Liouville vector field* χ_0 (i.e., satisfying $\mathcal{L}_{\chi_0}\omega_0 = \omega_0$) is given by

$$\chi_0 = \frac{1}{2} \sum_{j=1}^{n+1} (x_j \,\partial x_j + y_j \,\partial y_j).$$

Then on the boundary sphere $\mathbb{S}_{\varepsilon}^{2n+1}$, the 1-form

$$\alpha_0 = \iota_{\chi_0}\omega_0 = \frac{1}{2}\sum_{j=1}^{n+1}(x_j\,dy_j - y_j\,dx_j) = \frac{i}{4}\sum_{j=1}^{n+1}(z_j\,d\bar{z}_j - \bar{z}_j\,dz_j)$$

is a *contact form* (i.e., $\alpha_0 \wedge (d\alpha_0)^{\wedge n}|_{\mathbb{S}_{\varepsilon}^{2n+1}} > 0$). The codimension one plane distribution kernel $\xi_0 = \mathrm{Ker}(\alpha_0)$ is called the *standard contact structure* on $\mathbb{S}_{\varepsilon}^{2n+1}$.

The compatibility between contact structures and open books is defined as follows:

Definition 2.1 ([11])**.** We say that a contact structure $\xi = \mathrm{Ker}(\alpha)$ on M is *carried by* (or *supported by*) an open book (B, f) on M (where B is the binding), if the following conditions hold:

(i) $(B, \alpha|_{TB})$ is a contact manifold.
(ii) For every $t \in S^1$, the page $X = f^{-1}(t)$ is a symplectic manifold with symplectic form $d\alpha$.
(iii) If $\bar{X}$ denotes the closure of a page X in M, then the orientation of B induced by its contact form $\alpha|_{TB}$ coincides with its orientation as the boundary of $(\bar{X}, d\alpha)$.

The open book $\mathcal{OB}$ has been studied before, but since it is one of the main building blocks of the present paper and for completeness, here we discuss its important aspect:

Lemma 2.2. *The open book $\mathcal{OB}$ carries the standard contact structure ξ_0 on $\mathbb{S}_{\varepsilon}^{2n+1}$ inherited from the standard Stein structure on $\mathbb{B}_{\varepsilon}^{2n+2}$.*

Proof. The first condition of compatibility (Definition 2.1) immediately follows from [14] where they show that the restriction of α_0 is a contact form on Brieskorn manifolds, and so, in particular, on the binding K. To check the second one, consider the vector field

$$R = \frac{4i}{\varepsilon^2}\sum_{j=1}^{n+1} z_j\,\partial z_j = R_0 + R_1;\ R_0 = \frac{2i}{\varepsilon^2}\sum_{j=1}^{n+1}(z_j\,\partial z_j - \bar{z}_j\,\partial\bar{z}_j), R_1 = \frac{2i}{\varepsilon^2}\sum_{j=1}^{n+1}(z_j\,\partial z_j + \bar{z}_j\,\partial\bar{z}_j).$$

Observe that $R_0|_{\mathbb{S}_{\varepsilon}^{2n+1}}$ is the *Reeb vector field* of the contact form $\alpha_0|_{\mathbb{S}_{\varepsilon}^{2n+1}}$. (That is, we have $\alpha(R_0) = 1, \iota_{R_0}d\alpha_0 = 0$ on $\mathbb{S}_{\varepsilon}^{2n+1}$.) The flow of R is computed as

$$h_t(z) = (e^{4it/\varepsilon^2}z_1, ..., e^{4it/\varepsilon^2}z_{n+1})$$

which is a 1-parameter group of diffeomorphisms $h_t : \mathbb{C}^{n+1} \setminus Z(P) \to \mathbb{C}^{n+1} \setminus Z(P)$. Now consider the fibration $\Psi : \mathbb{C}^{n+1} \setminus Z(P) \to S^1$ given by

$$\Psi(z) = \frac{P(z)}{|P(z)|}.$$

Then h_t maps each fiber $\Psi^{-1}(y)$ diffeomorphically onto the fiber $\Psi^{-1}(e^{i\theta}y)$, and also there is a diffeomorphism $\Psi^{-1}(y) \cong \Theta^{-1}(y) \times \mathbb{R}$ as shown in Chapter 9 of [15]. Furthermore, h_t maps $\mathbb{S}_{\varepsilon}^{2n+1} \setminus K$ diffeomorphically onto itself for all t. Hence, we conclude that h_t maps each fiber $\Theta^{-1}(y)$ diffeomorphically onto the fiber $\Theta^{-1}(e^{i\theta}y)$, but this means, in particular, that the Reeb vector field $R_0|_{\mathbb{S}_{\varepsilon}^{2n+1}}$ is transverse to every page of the open book $\mathcal{OB}$ (note that R_1 does not live in $T\mathbb{S}_{\varepsilon}^{2n+1}$). So for any page F_θ, the rank of $d\alpha_0|_{F_\theta}$ is maximal which is equivalent to saying that $d\alpha_0$ is a symplectic form on F_θ.

For the third condition, on $T\mathbb{B}^{2n+2}_\varepsilon|_{\mathbb{S}^{2n+1}_\varepsilon}$ we compute $\omega_0(\chi_0, R_0) = 1$, so $\{\chi_0, R_0\}$ form a non-degenerate pair with respect to $\omega_0 = d\alpha_0$. Therefore, for a fixed page

$$F_\theta = \{z \,|\, \Theta(z) = e^{i\theta}\} \subset \mathbb{S}^{2n+1}_\varepsilon \setminus K$$

of the fibration Θ, the tangent bundle $T\mathbb{B}^{2n+2}_\varepsilon$ restricted to F_θ is decomposed as

$$T\mathbb{B}^{2n+2}_\varepsilon|_{F_\theta} = \langle \chi_0 \rangle \oplus T\mathbb{S}^{2n+1}_\varepsilon|_{F_\theta} = \langle \chi_0 \rangle \oplus \langle R_0 \rangle \oplus TF_\theta$$

where we use the fact that the Liouville vector field χ_0 is transverse to $\mathbb{S}^{2n+1}_\varepsilon$ (which is of contact type) and that R_0 is transverse to F_θ. This shows that $(F_\theta, \omega_0|_{F_\theta})$ is a symplectic submanifold of $(\mathbb{B}^{2n+2}_\varepsilon, \omega_0)$ and, in particular, that the orientation on F_θ given by $\omega_0|_{F_\theta}$ is inherited from the orientation on $\mathbb{B}^{2n+2}_\varepsilon$ given by ω_0.

Write $\alpha' = \alpha_0|_K$, $F = F_\theta$. To finish the proof, we need to check that the orientation on $\partial F = K$ given by the form $\alpha' \wedge (d\alpha')^{\wedge n-1}$ coincides with the one induced by the orientation on F given by the volume form $(d\alpha_0|_F)^{\wedge n}$: We showed above that the latter orientation on F is inherited from the one on $\mathbb{B}^{2n+2}_\varepsilon$ given by the standard Stein structure (J_0, ψ_0). Note that the orientation on $(\mathbb{S}^{2n+2}_\varepsilon, \xi_0)$ given by the volume form $\alpha_0 \wedge (d\alpha_0)^{\wedge n}$ is also coming from this Stein structure. Moreover, the orientation on $(K, \xi' := \mathrm{Ker}(\alpha')) \subset (\mathbb{S}^{2n+1}_\varepsilon, \xi_0)$ determined by $\alpha' \wedge (d\alpha')^{\wedge n-1}$ matches up with the one inherited (as a contact submanifold) from $(\mathbb{S}^{2n+1}_\varepsilon, \xi_0)$. Hence, the mentioned two orientations on K must coincide. □

2.2. **Positive stabilization of open books.** We first recall the plumbing or 2-Murasugi sum of two *contact open books* (i.e., open books carrying contact structures): Let (M_i, ξ_i) be two closed contact manifolds such that each ξ_i is carried by an open book (Σ_i, h_i) on M_i. Suppose that L_i is a properly embedded Lagrangian ball in Σ_i with Legendrian boundary $\partial L_i \subset \partial \Sigma_i$. By the Weinstein neighborhood theorem each L_i has a standard neighborhood N_i in Σ_i which is symplectomorphic to $(T^*D^n, d\lambda_{\text{can}})$ where $\lambda_{\text{can}} = \mathbf{pdq}$ is the canonical 1-form on $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(\mathbf{q}, \mathbf{p})$. Then the *plumbing* or *2-Murasugi sum* $(\mathcal{P}(\Sigma_1, \Sigma_2; L_1, L_2), h)$ of (Σ_1, h_1) and (Σ_2, h_2) *along* L_1 and L_2 is the open book on the connected sum $M_1 \# M_2$ with the pages obtained by gluing Σ_i's together along N_i's by interchanging $\mathbf{q}$-coordinates in N_1 with $\mathbf{p}$-coordinates in N_2, and vice versa. To define h, extend each h_i to $\tilde{h}_i$ on the new page by requiring $\tilde{h}_i$ to be identity map outside the domain of h_i. Then the monodrodmy h is defined to be $\tilde{h}_2 \circ \tilde{h}_1$. Without abuse of notation we will drop the "tilde" sign, and write $h = h_2 \circ h_1$.

The following terminology was given in [11]. We describe it in a slightly different way so that it fits into the notation of the present paper.

Definition 2.3 ([11])**.** Suppose that (Σ, h) carries the contact structure $\xi = \mathrm{Ker}(\alpha)$ on a $(2n+1)$-manifold M. Let L be a properly embedded Lagrangian n-ball in a page $(\Sigma, d\alpha)$ such that $\partial L \subset \partial \Sigma$ is a Legendrian $(n-1)$-sphere in the binding $(\partial \Sigma, \alpha|_{\partial \Sigma})$. Then the *positive* (or *standard*) *stabilization* $\mathcal{S}_{\mathcal{OB}}[(\Sigma, h); L]$ of (Σ, h) *along* L is the open book $(\mathcal{P}(\Sigma, \mathcal{D}(T^*S^n); L, \mathbf{D}), \delta \circ h)$ where $\mathbf{D} \cong D^n$ is any fiber in $\mathcal{D}(T^*S^n)$.

2.3. **Characterization of Lefschetz fibrations.** Here we recall the handle decomposition of Lefschetz fibrations as described in [12]: Let $\pi : W \to D^2 \subset \mathbb{C}$ be a given Lefschetz fibration with a regular fiber X^{2n} and monodromy h. Consider the base disk as $D^2 = \{z \in \mathbb{C} : |z| \leq 2\}$. We may assume that $0 \in D^2$ and the points on ∂D^2 are regular values and that all the critical values $\{\lambda_1, \lambda_2, .., \lambda_\mu\}$ of π are μ roots of unity. Such a π is called a *normalized Lefschetz fibration.* Since every Lefschetz fibration can be normalized, throughout the paper all Lefschetz fibrations will be assumed to be normalized. Define a Morse function $F : W \to [0,4] \subset \mathbb{R}$ given by $F(x) = |\pi(x)|^2$. Then outside of the set $F^{-1}(0) \cup F^{-1}(4)$, F has only nondegenerate critical points of index $n+1$ (see [1]). Since $|\lambda_i| = 1$ for all i, the map π has no critical values on the set $D_t = \{z \in \mathbb{C} : |z| \leq t\}$ for $t < 1$ and hence

$$F^{-1}([0,t]) = \pi^{-1}(D_t) \cong X \times D^2 \quad \text{for } t < 1.$$

On the other hand, for $t > 1$, $\pi^{-1}(D_t)$ is diffeomorphic to the manifold obtained from $X \times D^2$ by attaching μ handles of index $n+1$, via the attaching maps

$$\Phi_j : S^n \times D^{n+1} \to \partial(X \times D^2) = X \times S^1, \quad j = 1, 2, ..., \mu.$$

Let $\Phi'_j : \epsilon^{n+1} \to \nu$ be the framing of the j-th handle, where ϵ^k denotes the trivial bundle of rank k, and ν denotes the normal bundle of the attaching sphere $\Phi_j(S^n \times \{0\})$ in $\partial(X \times D^2)$.

Fact 2.4 ([12])**.** The embeddings Φ_j may be chosen so that for each $j = 1, 2, ..., \mu$ there exists z_j such that $\Phi_j(S^n \times \{0\}) \subset \pi^{-1}(z_j) \cong X$.

So, set $\phi_j : S^n \to X$ to be the embedding defined by restricting Φ_j to $S^n \times \{0\}$. Let ν_1 denote the normal bundle of $S^n \cong \phi_j(S^n)$ in X corresponding to the embedding ϕ_j, and consider ν as the normal bundle of $\phi_j(S^n)$ in $F^{-1}(1-\delta)$. Clearly, $\nu \cong \nu_1 \oplus \epsilon$ (as the normal bundle of X in W is trivial). Let τ denote the tangent bundle of S^n.

Fact 2.5 ([12])**.** For each $j = 1, 2, ..., \mu$, there exists a bundle isomorphism $\phi'_j : \tau \to \nu_1$ such that the framing Φ'_j of the $(n+1)$-handle corresponding to λ_j coincides with ϕ'_j. That is, Φ'_j is given by the composition

$$\epsilon^{n+1} \xrightarrow{\equiv} \tau \oplus \epsilon \xrightarrow{\phi'_j \oplus id} \nu_1 \oplus \epsilon \xrightarrow{\equiv} \nu.$$

Definition 2.6 ([12])**.** $S^n \cong \phi_j(S^n)$ is called a *vanishing cycle* of π. The bundle isomorphism $\phi'_j : \tau \to \nu_1$ is called a *normalization* of ϕ_j. The pair (ϕ_j, ϕ'_j) is called a *normalized vanishing cycle*.

Let $\mathcal{D}(TS^n) \subset \tau$ denote the closed tangent unit disk bundle of S^n. By the tubular neighborhood theorem and the canonical isomorphism $\mathcal{D}(T^*S^n) \cong \mathcal{D}(TS^n)$, we can apply the right-handed Dehn twist δ to a tubular neighborhood of $\phi_j(S^n)$ in X, and we can extend δ, by the identity, to a self-diffeomorphism of X which we denote by

$$\delta_{(\phi_j, \phi'_j)} : X \xrightarrow{\approx} X.$$

Up to smooth isotopy $\delta_{(\phi_j, \phi'_j)} \in \text{Diff}(X)$ depends only on the smooth isotopy class of the embedding ϕ_j and the bundle isomorphism ϕ'_j.

Definition 2.7 ([12])**.** $\delta_{(\phi_j, \phi'_j)}$ is called the *right-handed Dehn twist with center* (ϕ_j, ϕ'_j).

We will make use of the following theorem.

Theorem 2.8 ([12], [8])**.** *The Lefschetz fibration $\pi : W \to D^2$ is uniquely determined by a sequence of vanishing cycles $(\phi_1, \phi_2, ..., \phi_\mu)$ and a sequence of their normalizations $(\phi'_1, \phi'_2, ..., \phi'_\mu)$. The monodromy of the fibration is equal to*

$$\delta_\mu \circ \cdots \circ \delta_2 \circ \delta_1 \in \text{Diff}(X)$$

where $\delta_j = \delta_{(\phi_j, \phi'_j)}$ is the right-handed Dehn twist with center (ϕ_j, ϕ'_j).

Remark 2.9. Recall the right-handed Dehn twist $\delta : \mathcal{D}(T^*S^n) \to \mathcal{D}(T^*S^n)$ given explicitly before. With respect to the coordinates $(\mathbf{q}, \mathbf{p})$ on $\mathbb{R}^{2(n+1)}$ consider the canonical 1-form $\lambda_{can} = \mathbf{p} \cdot d\mathbf{q}$ on $\mathcal{D}(T^*S^n) \subset \mathbb{R}^{2(n+1)}$. Then one can compute

$$\delta^*\lambda_{can} = \lambda_{can} + |\mathbf{p}|d(g(|\mathbf{p}|))$$

which implies that the difference $\delta^*\lambda_{can} - \lambda_{can}$ is exact. Therefore, δ is a symplectomorphism of the symplectic manifold $(\mathcal{D}(T^*S^n), d\lambda_{can})$. As a result, if a regular fiber X of a Lefschetz fibration $\pi : W \to D^2$ equipped with a symplectic structure ω, then the monodromy h of π is a symplectomorphism of (X, ω). That is,

$$h = \delta_\mu \circ \cdots \circ \delta_2 \circ \delta_1 \in \mathrm{Symp}(X, \omega)$$

where $\delta_j = \delta_{(\phi_j, \phi'_j)}$ is the right-handed Dehn twist with center (ϕ_j, ϕ'_j) as in Theorem 2.8.

Notation 2.10. For our purposes it is convenient to define a notation for Lefschetz fibrations. Let the quadruple (π, W, X, h) denote the Lefschetz fibration $\pi : W \to D^2$ on W with a regular fiber X and the monodromy h. For instance, according to this notation we have $\mathcal{LF} = (\Pi, \mathbb{B}^{2n+2}_\varepsilon, \mathcal{D}(T^*S^n), \delta)$.

For completeness we give the following basic well-known fact as a definition:

Definition 2.11. Let (π, W, X, h) be any (normalized) Lefschetz fibration. The pairs $(\partial\pi^{-1}(0), \pi|_{\partial W})$ and (X, h) are both called the *induced open book* (or sometimes the *boundary open book*) on ∂W.

3. Positive stabilization of Lefschetz fibrations

Now we define a process on Lefschetz fibrations as a counterpart of positive stabilization on open books. We will use Weinstein handles introduced in [19]. Using the symplectization model near convex boundaries, these handles can be glued to symplectic manifolds along isotropic spheres to obtain new ones, and they give elementary symplectic cobordisms between contact manifolds. We will briefly explain them later.

Definition 3.1. Let (π, W, X, h) be a Lefschetz fibration which induces a contact open book on ∂W. Suppose that $L \subset (X, \omega)$ is a properly embedded Lagrangian n-ball with a Legendrian boundary $\partial L \subset \partial X$ on a page of the induced open book. Then the *positive stabilization* $\mathcal{S}_{\mathcal{LF}}[(\pi, W, X, h); L]$ of (π, W, X, h) *along* L is a Lefschetz fibration (π', W', X', h') described as follows:

(I) X' is obtained from X by attaching a Weinstein n-handle $H = D^n \times D^n$ along the Legendrian sphere $\partial L \subset \partial X$.

(II) $h' = \delta_{(\phi,\phi')} \circ h$ where $\delta_{(\phi,\phi')}$ is the right-handed Dehn twist with center (ϕ, ϕ') defined as follows: $\phi(S^n)$ is the Lagrangian n-sphere $S = D^n \times \{0\} \cup_{\partial L} L$ in the symplectic manifold $(X' = X \cup H, \omega')$ where ω' is obtained by gluing ω and standard symplectic form on H. If ν_1 denote the normal bundle of S in X', then the normalization $\phi' : \tau \to \nu_1$ is given by the bundle isomorphisms

$$\tau \xrightarrow[\phi_*]{\cong} TS \xrightarrow{\cong} TX'/TS = \nu_1.$$

Remark 3.2. W' is, indeed, diffeomorphic to W (see the proof of Theorem 3.3 below). Also in $h' = \delta_{(\phi,\phi')} \circ h$, we think of h as an element in Diff (X') by trivially extending over H. Moreover, the isomorphism $TS \to TX'/TS$ exists because S is Lagrangian in (X', ω') (the core $D^n \times \{0\}$ of H is Lagrangian). Finally, note that there is a strong analogy between $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$ and $\mathcal{S}_{\mathcal{LF}}[(\pi,W,X,h);L]$. On the one hand, we have

$$\mathcal{S}_{\mathcal{OB}}[(X,h);L] = (\mathcal{P}(X,\mathcal{D}(T^*S^n);L,\mathbf{D}),\delta\circ h)$$

which means that we are plumbing the open book (X,h) on a given manifold M with the open book $\mathcal{OB} = (\mathcal{D}(T^*S^n),\delta)$ on $\mathbb{S}^{2n+1}_\varepsilon$. Therefore, $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$ is an open book on the connected sum $M\#\,\mathbb{S}^{2n+1}_\varepsilon \approx M$. On the other hand, we may regard $\mathcal{S}_{\mathcal{LF}}[(\pi,W,\omega,\chi,X,h);L]$ as the result of (informally speaking) "Lefschetz plumbing" of (π,W,X,h) with $\mathcal{LF}$. Indeed, one can see that $\mathcal{S}_{\mathcal{LF}}[(\pi,W,X,h);L]$ is a Lefschetz fibration on the boundary connect sum $W\#_b\,\mathbb{B}^{2n+2}_\varepsilon \approx W$.

We are now ready to prove

Theorem 3.3. *Any positive stabilization $\mathcal{S}_{\mathcal{LF}}[(\pi,W,X,h);L]$ of a Lefschetz fibration (π,W,X,h) with a contact boundary open book induces the open book $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$. Conversely, if a contact open book (X,h) is induced by a Lefschetz fibration (π,W,X,h), then any positive stabilization $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$ of (X,h) is induced by $\mathcal{S}_{\mathcal{LF}}[(\pi,W,X,h);L]$.*

Proof. By definition of $\mathcal{S}_{\mathcal{LF}}$, the fiber X' is obtained from X by attaching $2n$-dimensional Weinstein n-handle H along $\partial L \subset \partial X$. Since every fiber over D^2 is gaining H, we are actually attaching a $(2n+2)$-dimensional handle

$$H' = H \times D^2 = D^n \times D^{n+2}$$

to W along $\partial L \subset \partial W$. Say the resulting manifold is $\widetilde{W}$, that is $\widetilde{W} = W \cup H'$. By extending the monodromy h (but calling it still h) trivially over H, we get an extended Lefschetz fibration $\widetilde{\pi} : \widetilde{W} \to D^2$ on $\widetilde{W}$, i.e., we get $(\widetilde{\pi},\widetilde{W},X',h)$. Note that $(\widetilde{\pi},\widetilde{W},X',h)$ is determined by Theorem 2.8. So far what we explained is the content of Stage (I) in Definition 3.1. In Stage (II), composing the monodromy h with $\delta_{(\phi,\phi')}$ corresponds to attaching an $(2n+2)$-dimensional handle H'' (so called a "Lefschetz handle") with index $n+1$ to $\widetilde{W}$ along the Lagrangian sphere S in the fiber (X',ω') of $(\widetilde{\pi},\widetilde{W},X',h)$. By Theorem 2.8, we know that $(\widetilde{\pi},\widetilde{W},X',h)$ extends over the handle H'' and we get the Lefschetz fibration $\mathcal{S}_{\mathcal{LF}}[(\pi,W,X,h);L] = (\pi',W',X',h')$ on the resulting manifold

$$W' = \widetilde{W} \cup H'' = W \cup H' \cup H''.$$

We immediately see that $\{H',H''\}$ form a canceling pair in the smooth category as the attaching sphere of H'' intersects the belt sphere of H' transversely once, and so W' is diffeomorphic to the original manifold W (indeed $W' = W\#_b\mathbb{B}^{2n+2}_\varepsilon$). Therefore, the open book (X',h') induced by $\mathcal{S}_{\mathcal{LF}}[(\pi,W,X,h);L]$ is an open on the original boundary ∂W. Next we need to see that (X',h') is indeed isomorphic (as an abstract open book) to $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$. To this end, first observe that in the plumbing $(\mathcal{P}(X,\mathcal{D}(T^*S^n);L,\mathbf{D}),\delta\circ h)$ we are embedding a tubular neighborhood $N(\mathbf{D})$ of $\mathbf{D}$ in $\mathcal{D}(T^*S^n)$ into the page X in such a way that the intersection $N(\mathbf{D})\cap\partial X$ is a tubular neighborhood of the Legendrian sphere $\partial L(\approx S^{n-1})$. Considering ∂L as the equator of the zero section $S^n \times \{\mathbf{0}\} \subset \mathcal{D}(T^*S^n)$, it is clear that the part $\mathcal{D}(T^*S^n) \setminus N(\mathbf{D})$ of $\mathcal{D}(T^*S^n)$ which is not mapped into X (during the plumbing) is the trivial bundle $\mathcal{D}(T^*D^n) \cong D^n \times D^n$. Note that the canonical symplectic structure on $\mathcal{D}(T^*S^n)$ restricts to the standard symplectic structure on $\mathcal{D}(T^*D^n)$ which implies that $\mathcal{D}(T^*S^n) \setminus N(\mathbf{D})$ is the Weinstein handle H glued to X along ∂L. Hence, the page of the open book $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$, that is, the resulting page of the plumbing, is X'. Also if we keep track of the vanishing cycle $S^n \times \{\mathbf{0}\} \subset \mathcal{D}(T^*S^n)$ in the above discussion, we immediately see that it corresponds to the Lagrangian n-sphere $S = C \cup_{\partial L} L$ where $C = D^n \times \{0\}$ is the (Lagrangian) core disk of the Weinstein handle H which means that the right-handed Dehn twist δ coincides with $\delta_{\phi,\phi'}$ described in Definition 3.1. Composing with h, we get $\delta \circ h = h'$ Thus, $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$ and (X',h') are isomorphic. This proves the first statement.

For the second statement we basically follow the same steps in a different order: If $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$ is a given stabilization, then by the above discussion we know that the new page is equal to $X' = X \cup H$. By assumption (X,h) is induced from (π, W, X, h). So by attaching $H' = H \times D^2$ (thickening of H) to W, each fiber of π gains the handle H, and we get $(\widetilde{\pi}, \widetilde{W}, X', h)$ on $\widetilde{W}$. Since $\delta = \delta_{(\phi,\phi')}$, $h' = \delta_{(\phi,\phi')} \circ h = \delta \circ h$. Therefore, we have $\mathcal{S}_{\mathcal{OB}}[(X,h);L] = (X',h')$. Moreover, composing δ with h (in the open book level) corresponds to attaching a Lefschetz handle H'' to $\widetilde{W}$ whose normalized vanishing cycle is (ϕ,ϕ'). Therefore, we obtain (π', W', X', h') on $W' = \widetilde{W} \cup H'' (\approx W)$. It is now clear that $\mathcal{S}_{\mathcal{OB}}[(X,h);L]$ is induced by $\mathcal{S}_{\mathcal{LF}}[(\pi, W, X, h);L]$. □

4. Exact Open Books

We will define exact open books as boundary open books induced by exact Lefschetz fibrations. To this end, recall that a contact manifold (M,α) is called *strongly symplectically filled* by a symplectic manifold (X,ω) if there exist a Liouville vector field χ of ω defined (at least) locally near $\partial X = M$ such that χ is transverse to M and $\iota_\chi \omega = \alpha$. Such a boundary is called *convex*. An *exact symplectic manifold* is a compact manifold X with boundary, together with a symplectic form ω and a 1-form α satisfying $\omega = d\alpha$, such that $\alpha|_{\partial X}$ is a contact form which makes ∂X convex. In such a case there is a Liouville vector field χ of ω such that $\iota_\chi \omega = \alpha$. We will write exact symplectic manifolds as triples of the form (X,ω,α). Also the pair (ω,α) (or sometimes the triple (ω,α,χ)) will be called an *exact symplectic structure* on X.

Let $\pi : E^{2n+2} \to D^2$ be a differentiable fiber bundle, denoted by (π, E), whose fibers are compact manifolds with boundary. The boundary of such an E consists of two parts: The vertical part $\partial_v E; = \pi^{-1}(\partial D^2)$, and the horizontal part $\partial_h E := \bigcup_{z \in D^2} \partial E_z$ where $E_z = \pi^{-1}(z)$ is the fiber over $z \in D^2$. The following definitions can be found in [16] where a more general setting is used.

Definition 4.1 ([16])**.** An *exact symplectic fibration* (π, E, ω, α) over the disk D^2 is a differentiable fiber bundle (π, E) equipped with a 2-form ω and a 1-form α on E, satisfying $\omega = d\alpha$, such that

(i) each fiber E_z with $\omega_z = \omega|_{E_z}$ and $\alpha_z = \alpha|_{E_z}$ is an exact symplectic manifold,

(ii) the following triviality condition near $\partial_h E$ is satisfied: Choose a point $z \in D^2$ and consider the trivial fibration $\tilde{\pi} : \tilde{E} := D^2 \times E_z \to D^2$ with the forms $\tilde{\omega}, \tilde{\alpha}$ which are pullbacks of ω_z, α_z, respectively. Then there should be a fiber-preserving diffeomorphism $\Upsilon : N \to \tilde{N}$ between neighborhoods N of $\partial_h E$ in E and $\tilde{N}$ of $\partial_h \tilde{E}$

in $\tilde{E}$ which maps $\partial_h E$ to $\partial_h \tilde{E}$, equals the identity on $N \cap E_z$, and $\Upsilon^* \tilde{\omega} = \omega$ and $\Upsilon^* \tilde{\alpha} = \alpha$.

Definition 4.2 ([16])**.** An *exact Lefschetz fibration* over D^2 is a tuple $(\pi, E, \omega, \alpha, J_0, j_0)$ which satisfies the following conditions:

(i) $\pi : E \to D^2$ is allowed to have finitely many critical points all of which lie in the interior of E.
(ii) π is injective on the set C of its critical points.
(iii) J_0 is an integrable complex structure defined in a neighborhood of C in E such that ω is a Kähler form for J_0.
(iv) j_0 is a positively oriented complex structure on a neighborhood of the set $\pi(C)$ in D^2 of the critical values.
(v) π is (J_0, j_0)-holomorphic near C.
(vi) The Hessian of π at any critical point is nondegenerate as a complex quadratic form, in other words, π has nondegenerate complex second derivative at each its critical point.
(vii) $(\pi, E \setminus C, \omega, \alpha)$ is an exact symplectic fibration over $D^2 \setminus \pi(C)$.

Remark 4.3. As pointed out in [16], one can find an almost complex structure on J on E agreeing with J_0 near C and a positively oriented complex structure j on D^2 agreeing with j_0 near $\pi(C)$ such that π is (J, j)-holomorphic and $\omega(\cdot, J\cdot)|_{\mathrm{Ker}\,(\pi_*)}$ is symmetric and positive definite everywhere. The existence of (J, j) is guaranteed by the fact that the space of such pairs (J, j) is always contractible, and in particular, always nonempty. Furthermore, once we fixed (J, j), we can modify ω by adding a positive 2-form on D^2 so that it becomes symplectic and tames J everywhere on E.

For completeness and our future use, let us summarize the discussion in the last remark and make additional observations about exact Lefschetz fibrations in the following lemma.

Lemma 4.4. *For any exact Lefschetz fibration $(\pi, E, \omega, \alpha, J_0, j_0)$, there exists an exact 2-form $\Omega = d\Lambda$ on the total space E and a pair (J, j) as in Remark 4.3 such that*

(i) *π is (J, j)-holomorphic and Ω is symplectic and tames J everywhere on E,*
(ii) *(E, Ω, Λ) is an exact symplectic manifold with convex boundary $(\partial E, \Lambda|_{\partial E})$,*
(iii) *each regular fiber $(E_z, \omega_z, \alpha_z)$ is an exact symplectic submanifold of (E, Ω, Λ).*

Proof. The first statement follows by Remark 4.3. More precisely, consider the positive 2-form $dr \wedge d\theta$ on D^2 where r is the radial and θ is the angular coordinates. Then it is standard to check that the form $\Omega = \omega + \pi^*(dr \wedge d\theta)$ is symplectic and tames J on E, and also that π is (J, j)-holomorphic. For the second one, we have

$$\Omega = \omega + \pi^*(dr \wedge d\theta) = d\alpha + \pi^*(d(rd\theta)) = d\alpha + d\pi^*(rd\theta) = d(\alpha + \pi^*(rd\theta)),$$

and so Ω is exact with a primitive $\Lambda = \alpha + \pi^*(rd\theta)$. Let $\mathcal{V}_z$ be the Liouville vector field of ω_z. By the local triviality condition near $\partial_h E$, these $\mathcal{V}_z$'s glue together (smoothly) and gives a Liouville vector field $\mathcal{V}$ for ω near $\partial_h E$. Now, consider the collar neighborhood N_ε of $\partial_v E$ in E which is projected (by π) onto an annular region of the form $(1-\varepsilon, 1] \times S^1 \subset D^2$ for $0 < \varepsilon < 1$. Consider the vector field $r\partial/\partial r$ in $TD^2|_{(1-\varepsilon,1]\times S^1}$. Taking ε small enough, we can find a lift $\mathcal{H}$ of $r\partial/\partial r$ which is a vector field in $TE|_{N_\varepsilon}$. Then $\pi_*(\mathcal{H}) = r\partial/\partial r$ by definition. Also note that $\mathcal{H}$ can never be tangent to the fibers of π in N_ε because if $\mathcal{H}|_p$

were tangent to the fiber E_z over the point $z = re^{i\theta} \in D^2$ (with $r \neq 0$) at some point $p \in E_z$, then this would give the contradiction

$$\mathbf{0} = \pi_*(\mathcal{H}|_p) = r\partial/\partial r|_{(r,\theta)} \neq \mathbf{0}.$$

(The first equality follows from the fact that π_* kills the fiber directions.) So, in particular, $\mathcal{H}$ is transverse to $\partial_v E$ and is pointing out from the boundary. Now set

$$\chi = \mathcal{V} + \mathcal{H}.$$

Clearly, χ is nonvanishing vector field in a collar neighborhood of $\partial E = \partial_h E \cup \partial_v E$. Also along the boundary ∂E, it is transverse and outward pointing. Moreover, it follows from standard computations that $\mathcal{L}_\chi \Omega = \Omega$ and $\iota\chi\Omega = \Lambda$. As a result, χ is a Liouville vector field of Ω and $\Lambda|_{\partial E}$ is a contact form which makes ∂E the convex boundary of the exact symplectic manifold (E, Ω, Λ).

For the last part, we simply observe that $\Omega|_{E_z} = \omega|_{E_z} = \omega_z$ and $\Lambda|_{E_z} = \alpha|_{E_z} = \alpha_z$ which shows that each regular fiber $(E_z, \omega_z, \alpha_z)$ is an exact symplectic submanifold of the total space (E, Ω, Λ). □

Notation 4.5. Since they will not be considered in our discussions, from now on we drop J_0 and j_0 from our notation. Also we will assume that $\omega = d\alpha$ is already modified as in the proof of Lemma 4.4 and that its Liouville vector field χ is already constructed and given to us. Furthermore, we also want to specify the regular fiber and the monodromy in our notation as before. Therefore, we introduce the following:

Let $(\pi, E, \omega, \alpha, \chi, X, h)$ denote an exact Lefschetz fibration over the disk D^2 with the following properties:

(i) The underlying smooth Lefschetz fibration is (π, E, X, h) with $h \in \mathrm{Symp}(X, \omega_X)$.
(ii) The Liouville vector field χ of ω is transverse to ∂E and outward pointing.
(iii) (E, ω, α) is an exact symplectic manifold with convex boundary $(\partial E, \alpha|_{\partial E})$.
(iv) $(X, \omega|_X, \alpha|_X)$ is an exact symplectic submanifold of (E, ω, α).

Note that any exact Lefschetz fibration over the disk D^2 admits such representation.

Definition 4.6. If an open book is induced by an exact Lefschetz fibration, then it is said to be an *exact open book*.

Theorem 4.7. *The exact open book (X, h) induced by a given exact Lefschetz fibration $(\pi, E, \omega, \alpha, \chi, X, h)$ caries the contact structure $\xi = \mathrm{Ker}(\alpha|_{\partial E})$ on ∂E.*

Proof. We need to show that all three conditions in Definition 2.1 hold. Assuming $(\pi, E, \omega, \alpha, \chi, X, h)$ is normalized (and so $z_0 = (0, 0)$ is a regular value), the binding of (X, h) is the boundary of the regular fiber $E_{z_0} = \pi^{-1}(z_0)$. We know that $(\partial E_{z_0}, \alpha_{z_0}|_{\partial E_{z_0}})$ is the convex boundary of $(E_{z_0}, \omega_{z_0}, \alpha_{z_0})$ which is an exact symplectic submanifold of (E, ω, α). Since $\alpha_{z_0}|_{\partial E_{z_0}} = (\alpha|_{E_{z_0}})|_{\partial E_{z_0}} = \alpha|_{\partial E_{z_0}}$, we conclude that $\alpha|_{\partial E_{z_0}}$ is a contact form on the binding ∂E_{z_0}, so the first condition follows.

For the second one, each regular fiber E_z of $(\pi, E, \omega, \alpha, \chi, X, h)$ is an exact symplectic submanifold of (E, ω, α) with the symplectic form $\omega_z = \omega|_{E_z} = d\alpha|_{E_z}$. In particular, any page X of the boundary open book (X, h) equips with the symplectic structure $d\alpha|_X$ as being a regular fiber of π.

To check the orientation condition, we need to specify the dimensions. Say E has dimension $2n + 2$, and so the page X and the binding B have dimensions $2n$ and $2n - 1$, respectively. For simplicity, write $\alpha' = \alpha|_B$ and $\omega' = \omega|_X (= d\alpha|_X)$. Let R' be the Reeb vector field of α' and let χ' be the Lioville vector field of ω' pointing out from B. To finish the proof, we need to check that at a given point $p \in \partial X = B$ the orientation on $T_p B$ given by the form $\alpha'_p \wedge (d\alpha'_p)^{\wedge n-1}$ coincides with the one induced by the orientation on $T_p X$ given by the volume form $(\omega'_p)^{\wedge n}$:

Consider the contact structure $\xi' := \text{Ker}(\alpha')$ which is a symplectic subbundle (with rank $2n-2$) of $\xi = \text{Ker}(\alpha)$, and the decomposition (see [10], for instance)

$$T_p B = \langle R'_p \rangle \oplus \xi'_p.$$

From their definitions, we have $\omega'(\chi', R') > 0$ which means that $\{\chi', R'\}$ is a nondegenerate pairing with respect to ω'. Also since they are both transverse to ξ', we get the decomposition

$$T_p X = \langle \chi'_p \rangle \oplus \langle R'_p \rangle \oplus \xi'_p.$$

Choose a symplectic basis $\{u_1, v_1, ..., u_{n-1}, v_{n-1}\}$ for the symplectic subspace (ξ'_p, ω'_p) giving the orientation on ξ'_p determined by $(\omega'_p)^{\wedge n-1}$, that is, we have

$$(\omega'_p)^{\wedge n-1}(u_1, v_1, ..., u_{n-1}, v_{n-1}) > 0.$$

Since (X, ω') is a symplectic manifold and $\omega'_p(\chi'_p, R'_p) > 0$, we get a symplectic basis

$$\{\chi'_p, R'_p, u_1, v_1, ..., u_{n-1}, v_{n-1}\}$$

for the symplectic space $(T_p X, \omega'_p)$ giving the orientation on $T_p X$ determined by $(\omega')^{\wedge n}$, equivalently, $(\omega'_p)^{\wedge n}(\chi'_p, R'_p, u_1, v_1, ..., u_{n-1}, v_{n-1}) > 0$. Then the induced orientation on the subspace $T_p B \subset T_p X$ is determined by the oriented basis

$$\{R'_p, u_1, v_1, ..., u_{n-1}, v_{n-1}\}$$

(χ'_p is outward pointing normal direction at $p \in B = \partial X$). Now, using the fact $\omega' = d\alpha'$, it is not hard to see that

$$\alpha'_p \wedge (d\alpha'_p)^{\wedge n-1}(R'_p, u_1, v_1, ..., u_{n-1}, v_{n-1}) > 0.$$

□

5. Isotropic Setups and Weinstein Handles

In this section we briefly recall the isotropic setups and Weinstein handles introduced in [19]. Using them we will continue to study exact Lefschetz fibrations in the next section.

5.1. **Isotropic setups.** Let $(M, \xi = \text{Ker}(\alpha))$ be a $(2n+1)$-dimensional contact manifold. Any subbundle η of the symplectic bundle $(\xi, d\alpha)$ has a symplectic orthogonal $\eta^{\perp'} \subset \xi$. Therefore, if Y is an isotropic submanifold of M, then $d\alpha|_Y = 0$ (as $\alpha|_Y = 0$), and so

$$TY \subset (TY)^{\perp'} \subset \xi$$

from which we obtain the quotient bundle,

$$CSN(M, Y) = (TY)^{\perp'} / TY$$

which is called the *conformal symplectic normal bundle* of Y. Moreover, if $N(M, Y)$ denotes the normal bundle of Y in M, then we have the decomposition

$$\begin{aligned} N(M, Y) = TM|_Y / TY &\cong TM|_Y / \xi_Y \oplus \xi_Y / (TY)^{\perp'} \oplus (TY)^{\perp'} / TY \\ &\cong \langle R_Y \rangle \oplus T^*Y \oplus CSN(M, Y) \end{aligned}$$

where R_Y is the Reeb vector-field R of α restricted to Y. If we further assume that Y is a sphere, then $\langle R_Y \rangle \oplus T^*Y$ has a naturally trivialization. Hence, as pointed out in [19], any given trivialization of $CSN(M,Y)$ determines a *framing* on Y (that is, the trivialization of the normal bundle $N(M,Y)$), and the latter can be used to perform a surgery on M along Y. Moreover, the resulting contact structure on the surgered manifold agrees with that of M away from Y. Such an elementary surgery can be achieved also by attaching a Weinstein handle by making use of "isotropic setups" which we recall next.

A quintuple of the form (P, ω, χ, M, Y) is called an *isotropic setup* if (P, ω) is a symplectic manifold, χ is a Liouville vector field, M is a hypersurface transverse to χ (hence a contact manifold), and Y is an isotropic submanifold of M. The following proposition is the basic tool enabling us to attach Weinstein handles.

Proposition 5.1 ([19]). *Let $(P_1, \omega_1, \chi_1, M_1, Y_1), (P_2, \omega_2, \chi_2, M_2, Y_2)$ be two isotropic setups. Suppose that a given diffomorphism $Y_1 \to Y_2$ is covered by a symplectic bundle isomorphism*

$$CSN(M_1, Y_1) \to CSN(M_2, Y_2).$$

Then there exist neighborhoods U_j of Y_j in M_j and an isomorphism of isotropic setups

$$\phi : (U_1, \omega_1|_{U_1}, \chi_1|_{U_1}, M_1 \cap U_1, Y_1) \to (U_2, \omega_2|_{U_2}, \chi_2|_{U_2}, M_2 \cap U_2, Y_2)$$

which restricts to the given map $Y_1 \to Y_2$, and induces the given bundle isomorphism.

5.2. **Weinstein handles.** Denote the coordinates on $\mathbb{R}^{2n+2} = \mathbb{R}^{2(n+1)}$ by

$$(x_0, y_0, x_1, y_1, ..., x_n, y_n)$$

and consider the standard symplectic structure on $\mathbb{R}^{2n+2}$ as

$$\omega_0 = \sum_{j=0}^{n} dx_j \wedge dy_j.$$

We will focus on two special Weinstein handles that we need for the present paper. Namely, let H_n and H_{n+1} be the $(2n+2)$-dimensional Weinstein handles in $\mathbb{R}^{2n+2}$ with indexes n and $n+1$, respectively. These handles are defined as follow: Consider

$$\chi_n = -\frac{x_0}{2}\frac{\partial}{\partial x_0} - \frac{y_0}{2}\frac{\partial}{\partial y_0} + \sum_{j=1}^{n}\left(-2x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right), \quad \chi_{n+1} = \sum_{j=0}^{n}\left(-2x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right)$$

which are the negative gradient vector fields of the Morse functions

$$f_n = \frac{x_0^2}{4} + \frac{y_0^2}{4} + \sum_{j=1}^{n}\left(x_j^2 - \frac{1}{2}y_j^2\right), \quad f_{n+1} = \sum_{j=0}^{n}\left(x_j^2 - \frac{1}{2}y_j^2\right)$$

respectively. We have the contractions $\alpha_k = \iota_{\chi_k}\omega_0$, for $k = n, n+1$, given as

$$\alpha_n = -\frac{x_0}{2}\,dy_0 + \frac{y_0}{2}\,dx_0 + \sum_{j=1}^{n}(-2x_j dy_j - y_j dx_j), \quad \alpha_{n+1} = \sum_{j=0}^{n}(-2x_j dy_j - y_j dx_j)$$

from which we compute that $\mathcal{L}_{\chi_k}\omega_0 = d(\iota_{\chi_k}\omega_0) = -\omega_0$. Therefore, χ_n, χ_{n+1} are both Liouville vector fields of ω_0. Next, consider the unstable manifold

$$E_-^k = \{x_0 = \cdots = x_n = y_0 = \cdots = y_{n-k} = 0\},$$

and the hypersurface $X_- = f_k^{-1}(-1)$ which is of contact type. The pull back of α_k on E_-^k is zero, and so the descending sphere

$$\mathcal{S}^{k-1} = E_-^k \cap X_-$$

is isotropic (Legendrian if $k = n+1$) in the contact manifold $(X_-, \alpha_k|_{X_-})$. Similarly, we have the stable manifold $E_+^{2n+2-k} = \{y_{n-k+1} = \cdots = y_n = 0\}$ and the hypersurface $X_+ = f_k^{-1}(1)$ intersecting each other along the ascending sphere

$$S^{2n+1-k} = E_+^{2n+2-k} \cap X_+$$

which is a submanifold of the contact manifold $(X_+, \alpha_k|_{X_+})$.

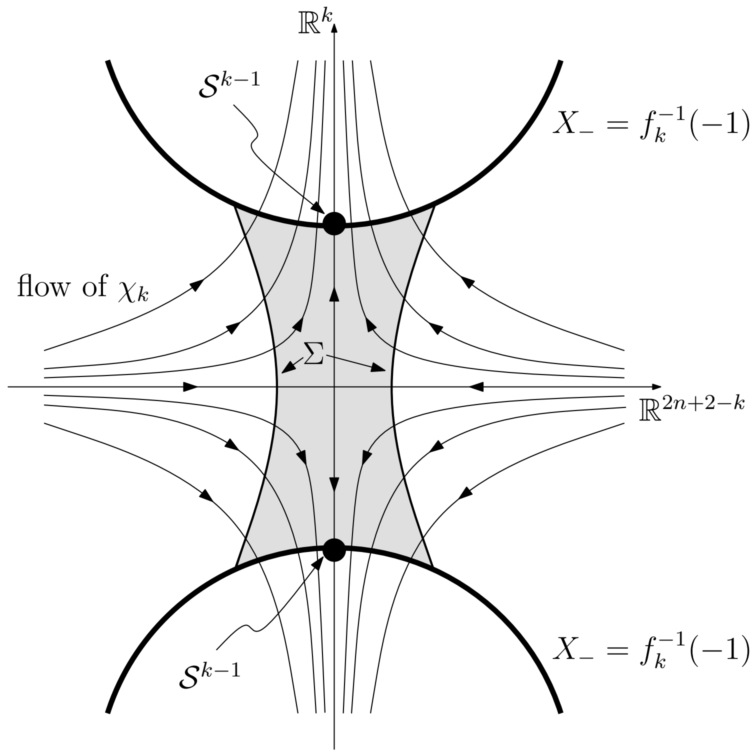

FIGURE 1. Weinstein handle H_k (shaded) and the flow of χ_k transverse to ∂H_k.

The Weinstein handle H_k is the region bounded by a neighborhood (which can be taken arbitrarily small) of the descending sphere $\mathcal{S}^{k-1}$ in X_- together with a connecting manifold $\Sigma \approx S^{2n+1-k} \times D^k$ depicted in Figure 1. It follows (see [19]) that we can choose Σ in such a way that χ_k is everywhere transverse to the boundary ∂H_k. Now we state the main theorem of [19] which tells us, in particular, when we can attach Weinstein handles and how the symplectic structure extends over the handle.

Theorem 5.2 ([19]). *Let Y be an isotropic sphere in the contact manifold M with a trivialization of $CSN(M,Y)$. Let M' be the manifold obtained from M by elementary surgery along Y. Then the elementary cobordism P from M to M' obtained by attaching* *a Weinstein handle to $M \times [0,1]$ along a neighborhood of Y carries a symplectic structure and a Liouville vector field which is transverse to M and M'. The contact structure induced on M is the given one, while that on M' differs from that on M only on the spheres where the surgery takes place.*

One important fact about gluing symplectic manifolds is not mentioned rigorously before (at least in [19]). For our purposes it is convenient to state it as a lemma:

Lemma 5.3. *Gluing two exact symplectic manifolds using an isomorphism of isotropic setups results in an exact symplectic manifold.*

Proof. Let $(P_1, \omega_1, \alpha_1)$ and $(P_2, \omega_2, \alpha_2)$ be two exact symplectic manifold, and suppose that (as in Proposition 5.1) there exists an isomorphism of isotropic setups

$$\phi : (U_1, \omega_1|_{U_1}, \chi_1|_{U_1}, M_1 \cap U_1, Y_1) \to (U_2, \omega_2|_{U_2}, \chi_2|_{U_2}, M_2 \cap U_2, Y_2)$$

which restricts to a given map $Y_1 \to Y_2$, and induces a given bundle isomorphism

$$CSN(M_1, Y_1) \to CSN(M_2, Y_2).$$

Let P be the manifold obtained by gluing $(P_1, \omega_1, \alpha_1)$ and $(P_2, \omega_2, \alpha_2)$ using the isomorphism ϕ. This exactly means that along the gluing region we are gluing ω_i's, χ_i's (and so α_i's) together using ϕ. Therefore, on the gluing region either of $(\omega_1, \alpha_1, \chi_1)$ or $(\omega_2, \alpha_2, \chi_2)$ defines an exact symplectic structure. Observe that on $P \setminus P_2$ (resp. on $P \setminus P_1$) the triple $(\omega_1, \alpha_1, \chi_1)$, (resp. $(\omega_2, \alpha_2, \chi_2)$) defines an exact symplectic structure. Hence, P equips with the exact symplectic structure which we write as $(\omega_1 \cup_\phi \omega_2, \alpha_1 \cup_\phi \alpha_2, \chi_1 \cup_\phi \chi_2)$. □

6. Convex Stabilizations

Our observation via isotropic setups and Weinstein handles is the fact that we can perform certain positive stabilizations, which will be called "convex stabilizations", on exact Lefschetz fibrations. Convex stabilizations will be defined explicitly at the end of the section where a summary of results and some corollaries are also presented in this new terminology. The main theorem of this section is

Theorem 6.1. *Any positive stabilization of an exact Lefschetz fibration along a properly embedded Legendrian disk is also an exact Lefschetz fibration.*

Proof. Let $(\pi, E, \omega, \alpha, \chi, X, h)$ be a given exact Lefschetz fibration. We have already checked in the proof of Theorem 3.3 that a positive stabilization $\mathcal{S}_{\mathcal{LF}}[(\pi, E, X, h); L]$ of the underlying Lefschetz fibration (π, E, X, h) is an another Lefschetz fibration which we denoted by (π', E', X', h'). So all we need to check is that the exact symplectic structure (ω, α, χ) extends over the handles H', H'' which we used to construct (π', E', X', h') so that we get an exact symplectic structure $(\omega', \alpha', \chi')$ on E'.

At this point one should ask why the Legendrian disk L given on a page X of the boundary exact open book (which carries $\xi = \text{Ker}(\alpha|_{\partial E})$ by Theorem 4.7) is also Lagrangian on the page $(X, d\alpha)$ (so that $\mathcal{S}_{\mathcal{LF}}[(\pi, E, X, h); L]$ makes sense). We can check this as follows: From the basic equality

$$d\alpha(u, v) = \mathcal{L}_u \alpha(v) - \mathcal{L}_v \alpha(u) + \alpha([u, v])$$

we immediately see that $d\alpha(u,v)=0$ for all $u,v\in TL$ (see Chapter III in [2] for a discussion on integrable submanifolds of contact structures). This shows that L is Lagrangian on the page $(X,d\alpha)$.

Consider the $2n$-dimensional Weinstein handle H (of index n) used in Definition 3.1 and in the proof of Theorem 3.3. Taking the coordinates on $\mathbb{R}^{2n}\supset H$ as $(x_1,y_1,...,x_n,y_n)$, we can symplectically embed H into H_n by the map

$$(x_1,y_1,...,x_n,y_n)\to(0,0,x_1,y_1,...,x_n,y_n).$$

Indeed, we can trivially fiber H_n over D^2 with fibers diffeomorphic to H by constructing it in a different way as follows: Our new model for H_n will be $H\times D^2$. Consider the standard symplectic form $\omega_H=\sum_{j=1}^n dx_j\wedge dy_j$ on H whose Liouville vector field χ_H, the corresponding Morse function f_H and the contraction $\alpha_H=\iota_{\chi_H}\omega_H$ are

$$\chi_H=\sum_{j=1}^n\left(-2x_j\frac{\partial}{\partial x_j}+y_j\frac{\partial}{\partial y_j}\right),f_H=\sum_{j=1}^n\left(x_j^2-\frac{y_j^2}{2}\right),\alpha_H=\sum_{j=1}^n\left(-2x_jdy_j-y_jdx_j\right).$$

Let (r,θ) be the radial and the angle coordinates on D^2-factor in $H\times D^2$. If pr_1 (resp. pr_2) denotes the projection onto H-factor (resp. D^2-factor), then, similar to the proof of Lemma 4.4, the modification

$$\omega^0:=pr_1^*(\omega_H)+pr_2^*(rdr\wedge d\theta)$$

is a symplectic form on the total space $H_n=H\times D^2$ of the fibration $pr_2:H_n\to D^2$, and indeed is equivalent to the standard symplectic form ω_0. Considering χ_H and $-r/2\,\partial/\partial r$ as vector fields in $T(H\times D^2)=TH\times TD^2$, it is straightforward to check that

$$\chi^0:=\chi_H-r/2\,\partial/\partial r$$

is the Liouville vector field of ω^0 (satisfying $\mathcal{L}_{\chi^0}\omega^0=-\omega^0$) which gives the contraction

$$\alpha^0:=\iota_{\chi^0}\omega^0=\alpha_H-r^2/2\,d\theta.$$

Note that χ_H is transverse to $\partial_h H_n=\partial H\times D^2$ and $-r/2\,\partial/\partial r$ is transverse to $\partial_v H_n=H\times S^1$, and so χ^0 is everywhere transverse to ∂H_n. It follows that each fiber

$$H_z:=pr_2^{-1}(z)\approx H\quad(z\in D^2)$$

is an $2n$-dimensional Weinstein handle (of index n) and equips with the exact symplectic form $\omega_z^0:=\omega^0|_{H_z}$ with the primitive $\alpha_z^0:=\alpha^0|_{H_z}$ and whose Liouville vector field $\chi_z^0:=\chi^0|_{H_z}$ is transverse to ∂H_z and satisfies $\iota_{\chi_z^0}\omega_z^0=\alpha_z^0$.

As a result, we obtain a trivial (no singular fibers) Lefschetz fibration (pr_2,H_n,H,id) over D^2. One should note that this is not an exact Lefschetz fibration because neither ∂H_n nor ∂H_z is convex, but it can be glued to an exact Lefschetz fibration along the convex part, which we will denote by $\partial^{CX}H_n$, of its boundary to construct a new exact Lefschetz fibration as we will see below. To describe $\partial^{CX}H_n$, we first observe that the boundary of H is decomposed into its convex and concave parts as

$$\partial H=\partial^{CX}H\cup\partial^{CV}H$$

where $\partial^{CX}H\approx\mathcal{S}^{n-1}\times D^n$ is the tubular neighborhood of descending sphere $\mathcal{S}^{n-1}$ in the hypersurface $f_H^{-1}(-1)$ from which χ_H points outward, and $\partial^{CV}H=\Sigma_H\approx S^{n-1}\times D^n$ is the connecting manifold from which χ_H points inward. Then we get the decomposition

$$\partial H_n=(\partial H\times D^2)\cup(H\times S^1)=(\partial^{CX}H\times D^2)\cup(\partial^{CV}H\times D^2)\cup(H\times S^1)$$

from which we deduce that

$$\partial^{CX} H_n = \partial^{CX} H \times D^2 \quad \text{and} \quad \partial^{CV} H_n = (\partial^{CV} H \times D^2) \cup (H \times S^1).$$

An easy way to understand this decomposition is given schematically in Figure 2.

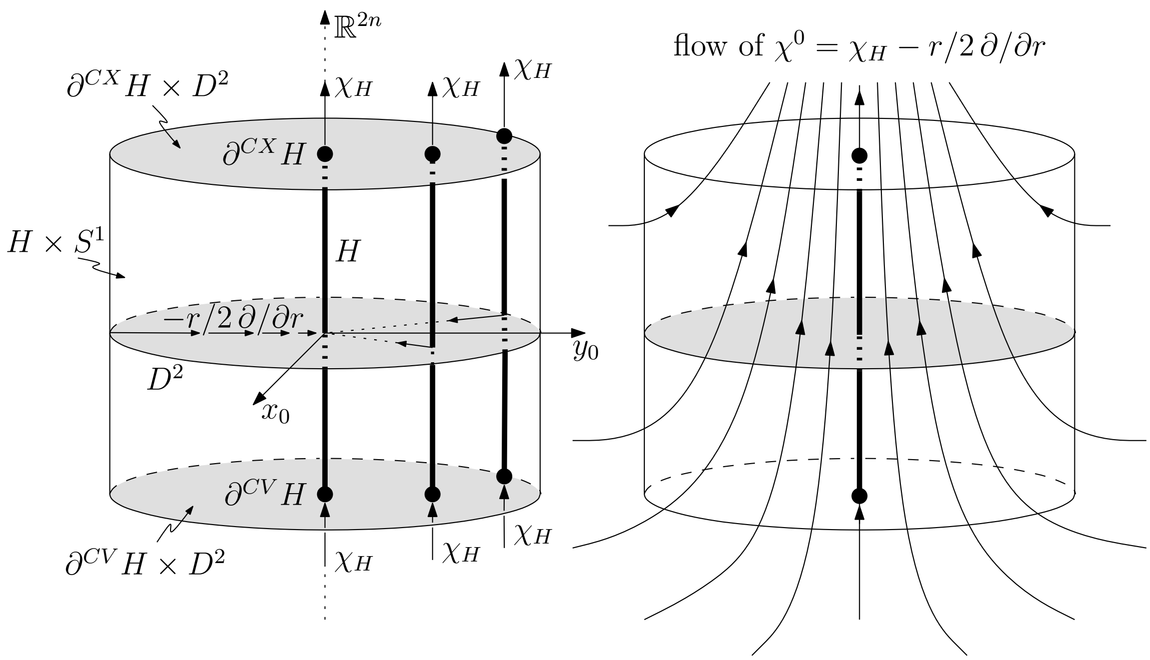

FIGURE 2. A schematic picture of the convex and concave parts of ∂H_n and the flow of $\chi^0 = \chi_H - r/2\,\partial/\partial r$ in $\mathbb{R}^{2n} \times \mathbb{R}^2$

Lemma 6.2. *The handle H' can be replaced by the handle H_n. Moreover, the exact symplectic structure (ω, α, χ) on E extends over the handle H_n.*

Proof. We will replace the handle $H' = H \times D^2$ used in the proof of Theorem 3.3 with the Weinstein handle $H_n = H \times D^2$. Here we do the replacement in such a way that the new fiber $\tilde{E}_z \approx X'$ over $z \in D^2$ is obtained from the fiber $X = E_z$ by attaching the Weinstein handle H_z along the Legendrian sphere $S_z := S^{n-1} \subset \partial E_z$ which we consider as a copy of the boundary ∂L of the Legendrian (and so Lagrangian) ball L of the stabilization. More precisely, we proceed as follows:

As S_z is Legendrian in $(\partial E_z, \alpha_z)$, its conformal symplectic normal bundle $CSN(\partial E_z, S_z)$ is zero (i.e., has rank zero). Similarly, the descending sphere $\mathcal{S}_z^{n-1}$ is Legendrian in $(\partial^{CX} H_z, \alpha_z^0)$ and so $CSN(\partial^{CX} H_z, \mathcal{S}_z^{n-1})$ is also zero. Therefore, by Proposition 5.1, we can find neighborhoods U_z of $\mathcal{S}_z^{n-1}$ in H_z and V_z of S_z in E_z and an isomorphism of isotropic setups

$$\phi_z : (U_z, \omega_z^0|_{U_z}, \chi_z^0|_{U_z}, \partial^{CX} H_z \cap U_z, \mathcal{S}_z^{n-1}) \to (V_z, \omega_z|_{V_z}, \chi_z|_{V_z}, \partial E_z \cap V_z, S_z)$$

which restricts to the map $f_z : \mathcal{S}_z^{n-1} \to S_z$ given in stage (I) of Definition 3.1. (Here f_z is the embedding of the attaching sphere of H_z.) Now using Theorem 5.2 we attach each

H_z to corresponding E_z using the isomorphism ϕ_z and obtain the new fiber X' equipped with exact symplectic structure

$$(\tilde{\omega}_z, \tilde{\alpha}_z, \tilde{\chi}_z) := (\omega_z \cup_{\phi_z} \omega_z^0, \alpha_z \cup_{\phi_z} \alpha_z^0, \chi_z \cup_{\phi_z} \chi_z^0).$$

(Note that $\omega_z \cup_{\phi_z} \omega_z^0 = d(\alpha_z \cup_{\phi_z} \alpha_z^0)$ and we use Lemma 5.3 to obtain this structure.)

Next, we fix a copy of $S_{z_0} \subset \partial E_{z_0}$ (with $z_0 \in \text{int}D^2$) of the Legendrian sphere ∂L in a fixed regular fiber E_{z_0}. Since z_0 is a regular value of π, we may assume that ∂E_{z_0} is the binding of the (exact) open book induced by π. Since this open book carries the contact structure $\xi = \text{Ker}(\alpha)$ on ∂E (which we know by Theorem 4.7), the binding $(\partial E_{z_0}, \xi_{z_0} := \text{Ker}(\alpha_{z_0}))$ is a contact submanifold manifold of $(\partial E, \xi)$, and so ξ_{z_0} is a subbundle of $(\xi|_{\partial E_{z_0}}, d\alpha|_{\partial E_{z_0}})$ and we have (see for instance [10])

$$T\partial E|_{\partial E_{z_0}} = T\partial E_{z_0} \oplus (\xi_{z_0})^{\perp'}$$

where $(\xi_{z_0})^{\perp'} = CSN(\partial E, \partial E_{z_0})$ is the symplectically orthogonal complement of ξ_{z_0} in $(\xi|_{\partial E_{z_0}}, d\alpha|_{\partial E_{z_0}})$. ($(\xi_{z_0})^{\perp'}$ is also called the conformal symplectic normal bundle of ∂E_{z_0} in ∂E.) The latter equality implies that $CSN(\partial E, \partial E_{z_0})$ can be identified with the classical normal bundle $N(\partial E, \partial E_{z_0})$ of ∂E_{z_0} in ∂E. But we know, by definition of open books, that the binding has a trivial normal bundle, so $CSN(\partial E, \partial E_{z_0}) = \partial E_{z_0} \times D^2$. Then from the inclusions $S_{z_0} \subset \partial E_{z_0} \subset \partial E$ we have

$$CSN(\partial E, S_{z_0}) = CSN(\partial E, \partial E_{z_0})|_{S_{z_0}} \oplus CSN(\partial E_{z_0}, S_{z_0}) = S_{z_0} \times D^2.$$

(Recall $CSN(\partial E_{z_0}, S_{z_0})$ is zero as S_{z_0} is Legendrian in ∂E_{z_0}.)

For $\mathcal{S}^{n-1}$, we follow not the same but similar lines: We fix a copy $\mathcal{S}_{z_0}^{n-1} \subset \partial^{CX} H_{z_0}$ in a fixed fiber H_{z_0} (with $z_0 \in \text{int}D^2$). The restriction of $\alpha_{z_0}^0$ onto $\partial^{CX} H_{z_0}$ is a contact form making $\partial^{CX} H_{z_0}$ convex, and $\mathcal{S}_{z_0}^{n-1}$ is Legendrian in $(\partial^{CX} H_{z_0}, \alpha_{z_0}^0|_{\partial^{CX} H_{z_0}})$. So we have

$$CSN(\partial^{CX} H_{z_0}, \mathcal{S}_{z_0}^{n-1}) = 0.$$

Also $(\partial^{CX} H_{z_0}, \alpha_{z_0}^0|_{\partial^{CX} H_{z_0}})$ is a contact submanifold manifold of $(\partial^{CX} H_n, \alpha^0|_{\partial^{CX} H_n})$. Then from the inclusions $\mathcal{S}_{z_0}^{n-1} \subset \partial^{CX} H_{z_0} \subset \partial^{CX} H_n$ we see that

$$CSN(\partial^{CX} H_n, \mathcal{S}_{z_0}^{n-1}) = CSN(\partial^{CX} H_n, \partial^{CX} H_{z_0})|_{\mathcal{S}_{z_0}^{n-1}} \oplus CSN(\partial^{CX} H_{z_0}, \mathcal{S}_{z_0}^{n-1}) = \mathcal{S}_{z_0}^{n-1} \times D^2.$$

Now we will show that all the above individual attachments are indeed pieces of the attachment of the Weinstein handle H_n to E along S_{z_0} by finding an isotropic setup which agrees with each individual fiber-wise gluing. To this end, note that we have the map $f_{z_0} : \mathcal{S}_{z_0}^{n-1} \to S_{z_0}$ given in stage (I) of Definition 3.1. Define the map

$$\Psi : CSN(\partial^{CX} H_n, \mathcal{S}_{z_0}^{n-1}) = \mathcal{S}_{z_0}^{n-1} \times D^2 \longrightarrow S_{z_0} \times D^2 = CSN(\partial E, S_{z_0})$$

by the rule

$$\Psi(p, z) = (f_{z_0}(p), z).$$

Clearly, Ψ is a bundle map and covers f_{z_0}, and so by Proposition 5.1, we can find neighborhoods U of $\mathcal{S}_{z_0}^{n-1}$ in H_n and V of S_{z_0} in E and an isomorphism of isotropic setups

$$\phi_n : (U, \omega^0|_U, \chi^0|_U, \partial^{CX} H_n \cap U, \mathcal{S}_{z_0}^{n-1}) \to (V, \omega|_V, \chi|_V, \partial E \cap V, S_{z_0})$$

which restricts to f_{z_0} and the bundle map Ψ. We may assume that $\partial^{CX} H_n \cap U = \partial^{CX} H_n$, that is, ϕ_n attaches H_n to E along the whole convex part $\partial^{CX} H_n = \partial^{CX} H \times D^2$ of its boundary. Now consider the boundaries

$$\partial^{CX} H_n = \partial^{CX} H_{z_0} \times D^2 \quad \text{and} \quad \partial_h E = \bigcup_{z \in D^2} \partial E_z = \partial E_{z_0} \times D^2.$$

For each $z \in D^2$, by attaching H_z to E_z using ϕ_z, we glue $\partial^{CX} H_{z_0} \times \{z\} \in \partial^{CX} H_n$ with $\partial E_{z_0} \times \{z\} \in \partial_h E$ and also we map $\mathcal{S}_z^{n-1}$ onto S_z by f_z. Therefore, attaching all H_z's to E_z's along ϕ_z's defines a smooth map $\mathcal{S}_{z_0}^{n-1} \times D^2 \longrightarrow S_{z_0} \times D^2$ which is identity on the D^2-factor and maps $\mathcal{S}_{z_0}^{n-1}$ onto S_{z_0} via f_{z_0}, and so it coincides with Ψ. Hence, we conclude that overall effect of attaching all H_z's to E_z's using ϕ_z's on E is equivalent to attaching Weinstein handle H_n to E using ϕ_n.

By Lemma 5.3 we know that the resulting manifold $\tilde{E} := E \cup_{\phi_n} H_n$ has an exact symplectic structure $(\tilde{\omega}, \tilde{\alpha}, \tilde{\chi})$ obtained by gluing those on E and H_n. In other words,

$$(\tilde{\omega}, \tilde{\alpha}, \tilde{\chi}) = (\omega \cup_{\phi_n} \omega^0, \alpha \cup_{\phi_n} \alpha^0, \chi \cup_{\phi_n} \chi^0).$$

Also, clearly, π extends over H_n and we get a Lefschetz fibration $\tilde{\pi} : \tilde{E} \to D^2$ with regular fiber X' and monodromy h (original h which is trivially extended over H). To check that $(\tilde{\omega}, \tilde{\alpha}, \tilde{\chi})$ restricts to $(\tilde{\omega}_z, \tilde{\alpha}_z, \tilde{\chi}_z)$ on each new regular fiber $\tilde{E}_z \approx X'$, we proceed as follows: For each $z \in D^2$, by taking U_z (resp. V_z) small enough, we can guarantee that the union

$$\bigcup_{z \in D^2} U_z \quad \left(\text{resp.} \bigcup_{z \in D^2} V_z\right)$$

lies in the collar neighborhood of $\partial^{CX} H_n$ (resp. $\partial_h E$) where we have the local triviality condition (as described in the definition of exact symplectic fibration). By using these local trivialities, we combine all the exact symplectic structures $(\omega_z \cup_{\phi_z} \omega_z^0, \alpha_z \cup_{\phi_z} \alpha_z^0, \chi_z \cup_{\phi_z} \chi_z^0)$ together, and surely the resulting structure must be $(\omega \cup_{\phi_n} \omega^0, \alpha \cup_{\phi_n} \alpha^0, \chi \cup_{\phi_n} \chi^0)$ because $(\omega_z, \alpha_z, \chi_z)$'s (resp. $(\omega_z^0, \alpha_z^0, \chi_z^0)$'s) patch together and give (ω, α, χ) (resp. $(\omega^0, \alpha^0, \chi^0)$). This completes the proof of Lemma 6.2. □

So far we have constructed an exact Lefschetz fibration $(\tilde{\pi}, \tilde{E}, \tilde{\omega}, \tilde{\alpha}, \tilde{\chi}, X', h)$ on

$$\tilde{E} := E \cup_{\phi_n} H_n,$$

in other words, we extended (ω, α, χ) over the handle H' by showing that H' can be replaced by H_n. Next, we want to extend $(\tilde{\omega}, \tilde{\alpha}, \tilde{\chi})$ over H'' by showing that H'' can be replaced by the Weinstein handle H_{n+1}.

Remark 6.3. Although Weinstein handles are attached along the convex part of their boundaries (according to the convention of present paper which coincides with the one in [19]), we actually need to reverse the direction of the Liouville vector field of the Weinstein handle when it is being attached to a convex boundary of a symplectic manifold. Otherwise it is impossible to match the Liouville directions of the symplectizations used in the gluing. Since we have attached H_n to E along the whole $\partial^{CX} H_n$ by matching $-\chi^0 = -\chi_H + r/2\, \partial/\partial r$ with χ, we have to now consider

$$\partial^{CV} H_n = (\partial^{CV} H \times D^2) \cup (H \times S^1)$$

as a subset of the convex part of the boundary $\partial \tilde{E}$.

Lemma 6.4. *The Lefschetz handle H'' can be replaced by the Weinstein handle H_{n+1}.*

Proof. Recall that H'' is attach to $\tilde{E}$ along the Lagrangian n-sphere S on a page $(X', d\tilde{\alpha}|_{X'})$ of the boundary exact open book (X', h) carrying the contact structure $\tilde{\xi} = \text{Ker}(\tilde{\alpha}|_{\partial\tilde{E}})$ on $\partial\tilde{E}$. Say $S \subset \tilde{E}_{\theta_0} (\approx X')$ for some $\theta_0 \in S^1 = \partial D^2$. From its construction (given in Definition 3.1) and the notation introduced above, S is the union

$$S = L \cup_{f_{\theta_0}} D$$

of the Lagrangian n-disk $L \in (E_{\theta_0}, \omega_{\theta_0} = d\alpha_{\theta_0})$ and the Lagrangian core disk $D\,(\approx D^n)$ of the $2n$-dimensional Weinstein handle $(H_{\theta_0}, d\alpha^0_{\theta_0})$. Note that $H_{\theta_0} \approx H$ is the fiber (over $\theta_0 \in S^1$) of the trivial fibration $H \times S^1$. By assumption L is Legendrian in $(\partial\tilde{E}, \tilde{\xi})$. On the other hand, the contact form $\tilde{\alpha}|_{\partial\tilde{E}}$ restricts to a contact form

$$\alpha_\sharp := (\tilde{\alpha}|_{\partial\tilde{E}})|_{H\times S^1} = (\iota_{-\chi^0}\omega^0)|_{H\times S^1} = -\alpha^0|_{H\times S^1} = \frac{d\theta}{2} - \alpha_H = \frac{d\theta}{2} + \sum_{j=1}^{n}(2x_j dy_j + y_j dx_j)$$

on a convex part $H \times S^1 \subset \partial\tilde{E}$. Observe that the core disk $D \subset H_{\theta_0} = H \times \{\theta_0\}$ is given by the set

$$\{x_1 = x_2 = \cdots = x_n = 0, \quad \theta = \theta_0(\text{constant})\},$$

and so clearly $\alpha_\sharp = 0$ on D which means that D is Legendrian in $(H \times S^1, \alpha_\sharp) \subset (\partial\tilde{E}, \alpha|_{\partial\tilde{E}})$. Therefore, the n-sphere S is also Legendrian in $(\partial\tilde{E}, \alpha|_{\partial\tilde{E}})$ which implies that $CSN(\partial\tilde{E}, S) = 0$. Moreover, we also have $CSN(\partial^{CX}H_{n+1}, \mathcal{S}^n) = 0$ as $\mathcal{S}^n$ is Legendrian in $(\partial^{CX}H_{n+1}, \alpha_{n+1}|_{\partial^{CX}H_{n+1}})$ by definition. Then by Proposition 5.1, we can find neighborhoods U' of $\mathcal{S}^n$ in H_{n+1} and V' of S in $\tilde{E}$ and an isomorphism of isotropic setups

$$\phi_{n+1} : (U', \omega_0|_{U'}, \chi_{n+1}|_{U'}, \partial^{CX}H_{n+1} \cap U', \mathcal{S}^n) \to (V', \tilde{\omega}|_{V'}, \tilde{\chi}|_{V'}, \partial\tilde{E} \cap V', S)$$

which restricts to the embedding $\phi : \mathcal{S}^n \to S$ determined by Definition 3.1. Now by Theorem 5.2 attaching H_{n+1} to $\tilde{E}$ using ϕ_{n+1} results in an exact symplectic manifold

$$E' := \tilde{E} \cup_{\phi_{n+1}} H_{n+1}$$

equipped with the exact symplectic data

$$(\omega', \alpha', \chi') = (\tilde{\omega} \cup_{\phi_{n+1}} \omega_0, \tilde{\alpha} \cup_{\phi_{n+1}} \alpha_{n+1}, \tilde{\chi} \cup_{\phi_{n+1}} \chi_{n+1}).$$

Again we may assume that $\partial^{CX}H_{n+1} \cap U' = \partial^{CX}H_{n+1}$, that is, ϕ_{n+1} attaches H_{n+1} to $\tilde{E}$ along the whole convex part $\partial^{CX}H_{n+1}$ of its boundary. Note that the step we just explained replaces H'' with the Weinstein handle H_{n+1}. From the bundle isomorphisms

$$\nu_1 \oplus \varepsilon \cong TS \oplus \varepsilon \cong T^*S \oplus (T\partial\tilde{E}/\tilde{\xi})|_S$$

we see that the framings on the normal bundle $N(\partial\tilde{E}, S)$ which are used to attach H'' and H_{n+1} coincide, and so attaching H'' and H_{n+1} are topologically the same. Therefore, we know by Theorem 2.8 that when we add H_{n+1} to $(\tilde{\pi}, \tilde{E}, \tilde{\omega}, \tilde{\alpha}, \tilde{\chi}, X', h)$, we can extend the underlying topological Lefschetz fibration $(\tilde{\pi}, \tilde{E}, X', h)$ over H_{n+1} and get the Lefschetz fibration

$$\mathcal{S}_{\mathcal{LF}}[(\pi, E, X, h); L] = (\pi', E', X', \delta_{(\phi,\phi')} \circ h)$$

where $\delta_{(\phi,\phi')}$ is the right-handed Dehn twist described in Definition 3.1. The proof of Lemma 6.4 is now complete. □

To be able to say that we have constructed an exact Lefschetz fibration

$$(\pi', E', \omega', \alpha', \chi', X', \delta_{(\phi,\phi')} \circ h)$$

on E', it remains to check that the exact symplectic structure $(\omega', \alpha', \chi')$ restricts to an exact symplectic structure on every new regular fiber E'_z. Note that this time we are not changing the diffeomorphism type of the regular fiber, that is $E'_z \approx \tilde{E}_z \approx X'$. The Weinstein handle H_{n+1} is attached to $\tilde{E}$ along the neighborhood

$$\partial \tilde{E} \cap V' \approx N(\tilde{E}_{\theta_0}, S) \times [0,1]$$

of the attaching sphere $S \subset \tilde{E}_{\theta_0}$ in $\partial \tilde{E}$ where we identify the interval $[0,1]$ with a closed arc in $S^1 \setminus \{pt\}$ such that $0 < \theta_0 < 1$. Consider the mapping torus

$$X' \times [0,1]/(x,0) \sim (h(x),1)$$

of the open book (X', h) on $\partial \tilde{E}$ and the inclusion

$$N(\tilde{E}_{\theta_0}, S) \times [0,1] \subset X' \times [0,1].$$

Observe that attaching H_{n+1} to $\tilde{E}$ along the attaching region $N(\tilde{E}_{\theta_0}, S) \times [0,1]$ results in a new mapping torus

$$X' \times [0,1]/(x,0) \sim ((\delta_{(\phi,\phi')} \circ h)(x),1)$$

for the open book $(X', \delta_{(\phi,\phi')} \circ h)$ on the new boundary $(\partial E', \xi' := \mathrm{Ker}(\alpha'|_{\partial E'}))$ obtained from the corresponding elementary (contact) surgery on $(\partial \tilde{E}, \tilde{\xi})$ along the Legendrian sphere S. To get this new mapping torus, we are just gluing two copies of X' equipped with the exact symplectic structure $(\tilde{\omega}_{X'}, d\tilde{\omega}_{X'}, \tilde{\chi}_{X'})$ using the symplectomorphism

$$\delta_{(\phi,\phi')} \circ h \in \mathrm{Symp}(X', \tilde{\omega}_{X'}).$$

Therefore, attaching H_{n+1} does not change the exact symplectic structures of regular fibers. But of course, it does change the structure of the Lefschetz fibration: Relative to $\tilde{\pi} : \tilde{E} \to D^2$, the new Lefschetz fibration $\pi' : E' = \tilde{E} \cup H_{n+1} \to D^2$ has one more critical point (and so one more singular fiber) located at the origin in the Weinstein handle H_{n+1}.

We conclude that $(\omega', \alpha', \chi')$ restricts to

$$(\omega'_z, \alpha'_z, \chi'_z) = (\tilde{\omega}_z, \tilde{\alpha}_z, \tilde{\chi}_z)$$

on each regular fiber $E'_z \approx \tilde{E}_z \approx X'$ of π'. Hence, we have an exact Lefschetz fibration $(\pi', E', \omega', \alpha', \chi', X', \delta_{(\phi,\phi')} \circ h)$ as claimed. This finishes the proof of Theorem 6.1. □

We have the following consequence of Theorem 6.1:

Corollary 6.5. *Any positive stabilization of an exact open book along a properly embedded Legendrian disk is also an exact open book.*

Proof. By definition if (X, h) is an exact open book, then there exist an exact Lefschetz fibration $(\pi, E, \omega, \alpha, \chi, X, h)$ which induces (X, h) on the boundary. Let L be any properly embedded Legendrian (and so Lagrangian) disk in (X, ω). Then by Theorem 3.3 we know that the stabilization $\mathcal{S}_{\mathcal{OB}}[(X,h); L]$ is induced by $\mathcal{S}_{\mathcal{LF}}[(\pi, E, X, h); L]$. Then Theorem 6.1 implies that there exists an exact Lefschetz fibration $(\pi', E', \omega', \alpha', \chi', X', h')$ whose underlying topological Lefschetz fibration is $\mathcal{S}_{\mathcal{LF}}[(\pi, E, X, h); L]$. In particular, $\mathcal{S}_{\mathcal{OB}}[(X,h); L]$ is induced by an exact Lefschetz fibration $(\pi', E', \omega', \alpha', \chi', X', h')$, and hence, it is exact by definition. □

After all, the following definitions make sense and fit into the frame very well.

Definition 6.6. (i) A *convex stabilization* $\mathcal{S}^{\mathcal{C}}_{\mathcal{LF}}[(\pi, E, \omega, \alpha, \chi, X, h); L]$ of an exact Lefschetz fibration $(\pi, E, \omega, \alpha, \chi, X, h)$ is defined to be the positive stabilization $\mathcal{S}_{\mathcal{LF}}[(\pi, E, X, h); L]$ where L is a properly embedded Legendrian disk on X.

(ii) A *convex stabilization* $\mathcal{S}^{\mathcal{C}}_{\mathcal{OB}}[(X, h); L]$ of an exact open book (X, h) is defined to be the positive stabilization $\mathcal{S}_{\mathcal{OB}}[(X, h); L]$ where L is a properly embedded Legendrian disk on X.

The theorem that we state next can be considered as the exact symplectic version of Theorem 3.3. It summarizes some of the results that we have shown in the language of convex stabilizations. The proof is a straight forward combination of previous statements and definitions, and so will be omitted.

Theorem 6.7. *$\mathcal{S}^{\mathcal{C}}_{\mathcal{LF}}[(\pi, E, \omega, \alpha, \chi, X, h); L]$ induces the (exact) open book $\mathcal{S}^{\mathcal{C}}_{\mathcal{OB}}[(X, h); L]$. Conversely, if an (exact) open book (X, h) is induced by $(\pi, E, \omega, \alpha, \chi, X, h)$, then any convex stabilization $\mathcal{S}^{\mathcal{C}}_{\mathcal{OB}}[(X, h); L]$ of (X, h) is induced by $\mathcal{S}^{\mathcal{C}}_{\mathcal{LF}}[(\pi, E, \omega, \alpha, \chi, X, h); L]$.* □

Combining the results we get so far, we know that a convex stabilization of an exact Lefschetz fibration produces an another exact Lefschetz fibration on a manifold which has the same diffeomorphism type with the original one. One can see that these manifolds are, indeed, symplectomorphic:

Theorem 6.8. *Let (E', ω', α') be the total space of $\mathcal{S}^{\mathcal{C}}_{\mathcal{LF}}[(\pi, E, \omega, \alpha, \chi, X, h); L]$, Then (E', ω', α') is symplectomorphic to (E, ω, α) In other words, the pair*

$$\{H_n, H_{n+1}\}$$

used in the construction of $\mathcal{S}^{\mathcal{C}}_{\mathcal{LF}}[(\pi, E, \omega, \alpha, \chi, X, h); L]$ is a symplectically canceling pair.

Proof. We have already observed in the proof of Theorem 3.3 that $\{H_n, H_{n+1}\}$ is a canceling pair in smooth category (as the belt sphere of H_n intersects the attaching sphere of H_{n+1} transversely once). Moreover, Lemma 3.6b in [6] (see also Lemma 3.9 in [18]) implies that two Weinstein handles form a symplectically canceling pair if they form a canceling pair in smooth category and their Morse-index difference is one. As a result, we conclude that $\{H_n, H_{n+1}\}$ is a canceling pair in symplectic category as well. □

As an immediate corollary, we have

Corollary 6.9. *Let ξ (resp. ξ') be the contact structure on ∂E (resp. $\partial E'$) induced by the exact symplectic structure of $(\pi, E, \omega, \alpha, \chi, X, h)$ (resp. $(\pi', E', \omega', \alpha', \chi', X', h') = \mathcal{S}^{\mathcal{C}}_{\mathcal{LF}}[(\pi, E, \omega, \alpha, \chi, X, h); L]$). Then $(\partial E, \xi)$ is contactomorhic $(\partial E', \xi')$.* □

Finally, as an application, we verify a well-known result for the class of exact open books and their convex stabilizations. Namely,

Corollary 6.10. *Let ξ be a contact structure carried by an exact open book (X, h). Then any convex stabilization $\mathcal{S}^{\mathcal{C}}_{\mathcal{OB}}[(X, h); L]$ of (X, h) carries ξ.*

Proof. By assumption, there exist an exact Lefschetz fibration $(\pi, E, \omega, \alpha, \chi, X, h)$ which induces (X, h) on the boundary. Note that, by Theorem 4.7, (ω, α, χ) induces ξ on ∂E. Theorem 6.7 implies that $\mathcal{S}^{\mathcal{C}}_{\mathcal{OB}}[(X, h); L]$ is induced by $\mathcal{S}^{\mathcal{C}}_{\mathcal{LF}}[(\pi, E, \omega, \alpha, \chi, X, h); L]$.

Moreover, again by Theorem 4.7, we know that $\mathcal{S}^{\mathcal{C}}_{\mathcal{OB}}[(X,h);L]$ carries the contact structure induced by the exact symplectic structure on $\mathcal{S}^{\mathcal{C}}_{\mathcal{LF}}[(\pi,E,\omega,\alpha,\chi,X,h);L]$. Now the proof follows from Corollary 6.9. □

References

[1] A. Andreotti and T. Frankel, *The second Lefschetz theorem on hyperplane section*, Global Analysis, Princeton University Press (1969), 1-20.

[2] D. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics 509, Springer-Verlag, 1976.

[3] E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1-14.

[4] K. Cieliebak and Y. Eliashberg *Symplectic geometry of Stein manifolds* (Incomplete draft) (2008)

[5] Deligne and Katz, *Groupes de Monodromie en Géométrie Algébrique*, Lecture Notes in Math. 340, Springer-Verlag.

[6] Y. Eliashberg, *Symplectic geometry of plurisubharmonic functions*, Gauge theory and symplectic geometry (Montreal, PQ, 1995), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 488, Kluwer Acad. Publ., Dordrecht, 1997, With notes by Miguel Abreu, pp. 49–67. MR 1461569 (98g:58055)

[7] Y. Eliashberg and M. Gromov, *Convex Symplectic Manifolds*, Proceedings of Symposia in Pure Mathematics, vol. 52, Part 2, 135–162 (1991).

[8] Deligne and Katz, *Groupes de Monodromie en Géométrie Algébrique*, Lecture Notes in Math. 340, Springer-Verlag.

[9] A. Dimca *Singularities and Topology of Hypersurfaces*, Springer-Verlag, 1992.

[10] H. Geiges, *Contact geometry*, Handbook of differential geometry. Vol. II, 315–382, Elsevier/North-Holland, Amsterdam, 2006.

[11] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing), Higher Ed. Press, (2002), pp. 405414. MR 2004c:53144

[12] A. Kas, *On the handlebody decomposition associated to a Lefschetz fibration*, Pasific Jornal of Math. (1980) vol. 89, No. 1, 89-104

[13] T. Lê Dũng, *Some remarks on the relative monodromy*, Real and complex singularities (Oslo 1976), Sijthoff and Noordhoff, Amsterdam, 1977, pp. 397-403.

[14] R. Lutz, C. Meckert *Structures de contact sur certaines sphères exotiques*, C. R. Acad. Sci. Paris Sér. A-B 282 (1976), no. 11, Aii, A591A593

[15] J. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton Univ. Press, 1968.

[16] P. Seidel, *A long exact sequence for symplectic Floer cohomology*, Topology 42 (2003), p. 1003-1063. (arXiv:math/0105186v2)

[17] I. Torisu, *Convex contact structures and fibered links in 3-manifolds*, Internat. Math. Res. Notices (2000), no. 9, 441454. MR MR1756943 (2001i:57039)

[18] O. Van Koert, *Lecture notes on stabilization of contact open books*, (arXiv:1012.4359v1)

[19] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. 20 (1991), no. 2, 241251.

Department of Mathematics, Michigan State University, Lansing MI, USA
E-mail address: akbulut@math.msu.edu

Max Planck Institute for Mathematics, Bonn, GERMANY
E-mail address: arikan@mpim-bonn.mpg.de