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Torsion in Gauge Theory

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1. Introduction

Let $P \longrightarrow X$ be a principal G-bundle over a closed simply connected 4-manifold, where $G = SU(2)$ or $SO(3)$, and $\mathcal{B}^*(P)$ be the gauge equivalence classes of connections in P. Recall that there is a weak homotopy equivalence:

$$\mathcal{B}^*(P) \simeq \mathrm{Map}^P(X, B_{SO(3)})$$

and the "universal" $SO(3)$-bundle $\xi \longrightarrow X \times \mathcal{B}^*(P)$ given by the evaluation map (cf.[DK]). In particular, for any finite complex K we have:

$$\begin{aligned}
[K, \mathcal{B}^*] &= \pi_0 Map\ (K, \mathrm{Map}^P(X, B_{SO(3)})) \\
&= \pi_0 Map\ (K \times X, B_{SO(3)})^P \\
&= [\, K \times X, B_{SO(3)}\,]^P \\
&= \{\, SO(3) \text{ bundles } \ \xi \longrightarrow K \times X \ \mid \ \xi \mid k \times X \cong P \ , \forall k \in K \,\}
\end{aligned}$$

Here $P \to K \times X$ denotes P pulled back by the projection $K \times X \to X$. The exponential P above means the set of maps that are homotopic to P on each slice $k \times X$ (by viewing view P as a map). In particular

$$\pi_k(\mathcal{B}^*) = \left\{ \begin{array}{c} \xi \\ \downarrow \\ S^k \times X \end{array} \ \mid \ \xi|_{s_0 \times X} = P \right\}$$

where s_0 denotes the base point of S^k. The last expression is just the isomorphisim class of bundles over $S^k \times X$ which restrict to a bundle isomorphic to P on each slice and equal to P on a particular slice (since $\pi_k(\mathcal{B}^*)$ are the homotopy classes of based maps). So we can identify $\pi_k(\mathcal{B}^*)$ as the number of different bundles on $D^k \times X$ restricting to the fix bundle $P \to S^{k-1} \times X$. The group structure is induced by the obvious way of composing two copies of $D^k \times X$ along their face $D^{k-1} \times X$. Now we can write

$$D^k \times X = S^{k-1} \times X \cup (k - \text{cell}) \cup (k + 2 - \text{cells}) \cup (k + 4 - \text{cell})$$

There are no indeterminacy's in applying the obstruction theory to count the bundles. For $i = k, k+2, k+4$, let $(D^k \times X)^{(i)}$ denote the i - skeleton of $D^k \times X$, and define

$$\begin{aligned} B_i &= \text{the number of different bundles on } S^{k-1} \times X \cup (D^k \times X)^{(i)} \\ &\quad \text{restricting to the fixed bundle } P \text{ on } S^{k-1} \times X \\ A_i &= \text{the number of different ways to extending a given bundle on} \\ &\quad S^{k-1} \times X \cup (D^k \times X)^{(i-2)} \text{ to } S^{k-1} \times X \cup (D^k \times X)^{(i)} \end{aligned}$$

Let $\theta_i : B_{i-2} \to O_i$ be the the obstruction to extending to the i-skeleton where

$$A_i = H^i(D^k \times X, S^{k-1} \times X; \pi_i B_{SO(3)}) = H^{i-k}(X; \pi_{i-1} SO(3))$$

$$O_i = H^i(D^k \times X, S^{k-1} \times X; \pi_{i-1} B_{SO(3)}) = H^{i-k}(X; \pi_{i-2} SO(3))$$

For example $\pi_k(\mathcal{B}^*) = B_{k+4}$. $\pi_k(\mathcal{B}^*)$ contains at least the "trivial" element P. A_i and O_i are possibly nonzero when $i = k, k+2, k+4$, so $\pi_k(\mathcal{B}^*)$ is a subset of

$$\mathcal{A}_k(X) := \pi_{k-1} SO(3) \oplus H^2(X; \pi_{k+1} SO(3)) \oplus \pi_{k+3} SO(3)$$

and the obstructions to realizing these bundles lie in

$$\mathcal{O}_k(X) := \pi_{k-2} SO(3) \oplus H^2(X; \pi_k SO(3)) \oplus \pi_{k+2} SO(3)$$

In particular, all $\pi_k(\mathcal{B}^*)$ are finite if $k > 4$. Recall that we have (e.g.[AMR])

$$\pi_1(\mathcal{B}^*(P)) = \begin{cases} Z_2 & \text{if } \left\{ \begin{array}{l} w_2(P) = w_2(X) \\ c_2(\bar{P}) = 0 \bmod 2 \end{array} \right\} \\ 0 & \text{otherwise} \end{cases}$$

Proposition ([A]) : If $\pi_1(\mathcal{B}^*) = 0$, $p > 3$ a prime, then $\pi_4(\mathcal{B}^*)_{(p)} = \mathbb{Z}$ and

$$\pi_2(\mathcal{B}^*(P)) = \begin{cases} H^2(X;\mathbb{Z}) \oplus \mathbb{Z}_2 & \text{if } w_2(P) = w_2(X) \\ H^2(X;\mathbb{Z}) & \text{otherwise} \end{cases}$$

$$\pi_k(\mathcal{B}^*)_{(3)} = \begin{cases} \mathbb{Z}_3 & \text{if } k = 3 \text{ or } 6, \text{ and } p_1(P) = \bmod 3 \\ 0 & \text{if } k = 3 \text{ or } 6, \text{ and } p_1(P) \neq \bmod 3 \\ H^2(X, \mathbb{Z}_3) & \text{if } k = 5 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \text{if } k = 7 \text{ and } p_1(P) = \bmod 3 \\ \mathbb{Z}_3 & \text{if } k = 7 \text{ and } p_1(P) \neq \bmod 3 \end{cases}$$

$$\pi_k(\mathcal{B}^*)_{(p)} = \begin{cases} 0 & \text{if } 4 < k < 2p-3 \\ \mathbb{Z}_p & \text{if } k = 2p-3 \\ H^2(X, \mathbb{Z}_p) & \text{if } k = 2p-1 \\ \mathbb{Z}_p & \text{if } k = 2p+1 \\ 0 & \text{for other } 2p-3 < k < 4p-6 \end{cases}$$

where the subscript (p) denotes the p-primary component. The only higher homotopy calculation that was not covered in [A] is $\pi_6(\mathcal{B}^*)_{(3)}$ and $\pi_7(\mathcal{B}^*)_{(3)}$; these will be calculated later in this paper. Recall that rational homology of $H^*(\mathcal{B}^*(P);\mathbb{Q})$ is a polynomial algebra [DK].

$$H^*(\mathcal{B}^*(P);\mathbb{Q}) = \mathbb{Q}[\mu_0(1), \mu_2(a_1), ..., \mu_2(a_n)]$$

where $\{a_1, ..., a_n\}$ is a set of generators of $H_2(X)$, and $\mu_i : H_i(X) \to H^{4-i}(\mathcal{B}^*)$ is the slant product map with $-p_1(\xi)/4$. In particular $\mu_0(1)$ is the Pontryagin class of the base point fibration, $\xi_0 \longrightarrow \mathcal{B}^*(P)$, which is obtained by restricting ξ to a slice. These classes have been used to define Donaldson polynomial invariants.

Hence it is natural to ask whether $H^*(\mathcal{B}^*(P);\mathbb{Z})$ contains torsion classes; and try to use them to define new invariants beyond Donaldson polynomials. For example, when $w_2(P)$ is characteristic and $c_2(P)$ is even then $H_1(\mathcal{B}^*(P);\mathbb{Z}) = \mathbb{Z}_2$. This 2-torsion gave the torsion invariant of [D], which was further investigated in [FS]. In this paper we will discuss the odd torsion of $H_*(\mathcal{B}^*(P);\mathbb{Z})$.

The restriction map $X \to X - int(B^4) \simeq \vee S^2$ gives the usual fibration:

$$\Omega^3_k SO(3) \longrightarrow \mathrm{Map}^P(X, BSO(3)) \longrightarrow Map(\vee S^2, SO(3))$$

Also by considering the restriction $X \to$ point, we get the base point fibration:

$$SO(3) \longrightarrow \mathrm{Map}^P_*(X, BSO(3)) \longrightarrow \mathrm{Map}^P(X, BSO(3))$$

where the $*$ indicates the base point preserving maps. If we are interested in computing odd torsion in homology groups we can replace $SO(3)$ by $SU(2)$, because the fibration $K(\mathbb{Z}_2, 1) \to BSU(2) \to BSO(3)$ induces the fibration:

$$K(\mathbb{Z}_2, 1) \to \mathrm{Map}^P(X, BSU(2)) \xrightarrow{\pi} \mathrm{Map}^P(X, BSO(3))$$

and the projection π induces isomorphisim in cohomology groups with $\mathbb{Z}_{2m+1}$ coefficients. Hence we get the following fibrations (horizontal and vertical maps):

$$\begin{array}{ccccc} & & S^3 & & \\ & & \downarrow & & \\ \Omega^3_k S^3 & \longrightarrow & \mathrm{Map}_*(X, BS^3) & \longrightarrow & \prod \Omega^1 S^3 \\ & & \downarrow & & \\ & & \mathrm{Map}(X, BS^3) & & \end{array}$$

Masbaum showed that [M] for primes $p > 3$ the horizontal fibration is a product

$$\mathrm{Map}_*(X, BS^3) \sim_p \Omega^3_k S^3 \times \prod \Omega^1 S^3$$

over the prime p. Homology of $\prod \Omega^1 S^3$ is torsion free, but the homology of $\Omega^3_k S^3$ consists of entirely torsion cycles. So we might hope to relate the induced torsion homology of Map$_*(X, BS^3)$ by the Gysin sequence to the homology of Map(X, BS^3). Unfortunately this is not immediate. Instead, we will take a more direct approach: We will produce torsion cycles by using secondary cohomology operations. This last splitting was also pointed out to us by F.Cohen.

2. Postnikov Tower of BU(2) at Odd Primes and Associated Cohomology Operations

Let p be an odd prime. In [T] the p primary components of homotopy groups of S^3 up to dimension $2p^2-2$ are tabulated as:

$$\pi_r(S^3)_{(p)} = \begin{cases} Z_p & \text{if } \left\{ \begin{array}{ll} r = 2p & \\ r = i(2p-2)+1 & i = 2,3,..p \\ r = i(2p-2)+2 & i = 2,3,..p \end{array} \right\} \\ 0 & \text{for all other } \ r < (p+1)(2p-2) \end{cases}$$

Also $\pi_r(S^3)_{(3)} = Z_3$ for $16 \leq r \leq 22$

Since $\pi_{r+1}BU(2) = \pi_r(S^3)$, mod p Postnikov tower of $BU(2)$ is in the form:

$$\begin{array}{rcl}
 & X_{p(2p-2)+3} = X_{(p+1)(2p-2)} & \\
 & \downarrow & \\
K(\mathbb{Z}_p, p(2p-2)+2) \longrightarrow & X_{p(2p-2)+2} \quad \xrightarrow{k_{p(2p-2)+4}} & K(\mathbb{Z}_P, p(2p-2)+4) \\
 & \downarrow & \\
K(\mathbb{Z}_p, (p-1)(2p-2)+3) \longrightarrow & X_{p(2p-2)+1} \quad \xrightarrow{k_{p(2p-2)+3}} & K(\mathbb{Z}_P, p(2p-2)+3) \\
 & \downarrow & \\
 & \cdots & \\
 & \downarrow & \\
K(\mathbb{Z}_p, i(2p-2)+2) \longrightarrow & X_{i(2p-2)+2} \quad \xrightarrow{k_{i(2p-2)+4}} & K(\mathbb{Z}_P, i(2p-2)+4) \\
 & \downarrow & \\
K(\mathbb{Z}_p, (i-1)(2p-2)+3) \longrightarrow & X_{i(2p-2)+1} \quad \xrightarrow{k_{i(2p-2)+3}} & K(\mathbb{Z}_P, i(2p-2)+3) \\
 & \downarrow & \\
 & \cdots & \\
 & \downarrow & \\
K(\mathbb{Z}_p, 2(2p-2)+2) \longrightarrow & X_{2(2p-2)+2} \quad \xrightarrow{k_{2(2p-2)+4}} & K(\mathbb{Z}_P, 2(2p-2)+4) \\
 & \downarrow & \\
K(\mathbb{Z}_p, 2p+1) \longrightarrow & X_{2(2p-2)+1} \quad \xrightarrow{k_{2(2p-2)+3}} & K(\mathbb{Z}_P, 2(2p-2)+3) \\
 & \downarrow & \\
BU(2)_{(p)} \xrightarrow{f} & X_{2p} \quad \xrightarrow{k_{2p+2}} & K(\mathbb{Z}_p, 2p+2)
\end{array}$$

where X_n approximates the n skeleton of $BU(2)_{(p)}$ and $X_{2p} = K(\mathbb{Z},2) \times K(\mathbb{Z},4)$, and $f = (c_1, c_2)$ is given by Chern classes. In [A] k_{2p+2} was computed to be:

$$k_{2p+2} = St_p^1(i_4) - i_4 s_{p-1}(i_2, i_4)$$

where s_{p-1} is the Newton polynomial satisfying $s_{p-1}(\sigma_1, \sigma_2) = x_1^{p-1} + x_2^{p-1}$ with $\sigma_1 = x_1 + x_2$ and $\sigma_2 = x_1 x_2$, and St_p^i is the Steenrod operation (c.f.[DFN]).

$$St_p^i : H^k(X; \mathbb{Z}_p) \to H^{k+2i(p-1)}(X; \mathbb{Z}_p)$$

For completeness let us check that f pulls back this class to zero. f pulls back i_2 and i_4, to c_1 and c_2. By the splitting principle we may assume that the universal bundle is the sum of line bundles $L = L_1 \oplus L_2$. Call $x_i = c_1(L_i)$, then

$$St_p^1(c_2) = St_p^1(x_1 x_2) = x_1^p x_2 + x_1 x_2^p = x_1 x_2 (x_1^{p-1} + x_2^{p-1}) = c_2 s_{p-1}(c_1, c_2).$$

By specializing $p = 3$, we can give a more concrete description of the tower :
Lemma 2.1 : The three stage Mod 3 Postnikov tower of $BU(2)$ is given by :

$$\begin{array}{ccccc}
K(\mathbb{Z}_3,10) & \xrightarrow{j} & X_{10} & \xrightarrow{k_{12}} & K(\mathbb{Z}_3,12) \\
 & & \pi_9 \downarrow & & \\
K(\mathbb{Z}_3,7) & \xrightarrow{j} & X_9 & \xrightarrow{k_{11}} & K(\mathbb{Z}_3,11) \\
 & & \pi_6 \downarrow & & \\
BU(2)_{(3)} & \xrightarrow{f} & X_6 & \xrightarrow{k_8} & K(\mathbb{Z}_3,8)
\end{array}$$

$X_6 = K(\mathbb{Z},2) \times K(\mathbb{Z},4)$, and $f = (c_1, c_2)$ is given by the Chern classes. Also if i_n denotes the corresponding fundamental classes of $H^n(K(\mathbb{Z},n);\mathbb{Z}_3)$ or $H^n(K(\mathbb{Z}_3,n);\mathbb{Z}_3)$, and δ the mod 3 Bokstein operation then:

$$k_8 = St_3^1(i_4) - i_4 i_2^2 - i_4^2$$
$$j \circ k_{11} = St_3^1(i_7)$$
$$k_{12} = r \circ \pi_9, \text{ where } r : X_9 \to K(\mathbb{Z}_3, 12), \text{ with } r \circ j = St_3^1 \delta(i_7)$$

Proof . $k_8 = St_3^1(i_4) - i_4 i_2^2 + 2i_4^2$ is calculated above. The next nonzero 3 primary homotopy group above 7 is $\pi_{10} BU(2)_{(3)} = \mathbb{Z}_3$. This gives the second k-invariant $k_{11} \in H^{11}(X_9, \mathbb{Z}_3)$, where X_9 is the total space of the pulled-back universal fibration, by k_8. We have a fibration $F_{10} \to BU(2) \to X_9$, with F_{10} is 9-connected and k_{11} is the transgression of $H^{10}(F_{10}; \mathbb{Z}_3)$. From Serre exact sequence of the fibration k_{11} is the unique element of $H^{11}(X_9; \mathbb{Z}_3) = \mathbb{Z}_3$.

Consider the spectral sequence of $K(\mathbb{Z}_3, 7) \to X_9 \to X_6$

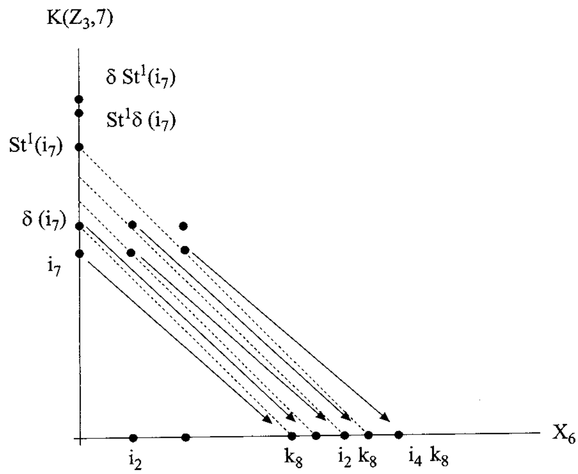

Figure 1.

By construction the fundamental class $i_7 \in H^7(K(\mathbb{Z}_3,7);\mathbb{Z}_3)$ transgresses to k_8. As a stable operation $St_3^1(i_7)$ transgresses to $St_3^1(k_8)$ by $d_{12} : E_{12}^{0,11} \to E_{12}^{12,0}$. By substitution:

$$\begin{aligned} St_3^1(k_8) &= St_3^1 St_3^1(i_4) - St_3^1(i_4 i_2^2) - St_3^1(i_4^2) \\ &= -St_3^2(i_4) - (St_3^1(i_4) i_2^2 - i_4 i_2^4) + i_4 St_3^1(i_4) \\ &= (-i_4^3 + i_4 St_3^1(i_4) - i_4^2 i_2^2) - (i_2^2 St_3^1(i_4) - i_2^4 i_4 - i_2^2 i_4^2) \\ &= i_4 k_8 - i_2^2 k_8 \end{aligned}$$

Last two terms are zero since they are killed in an earlier stage by the images of $i_4 \otimes i_7$ and $i_2^2 \otimes i_7$ under $d_8 : E_8^{4,7} \to E_8^{12,0}$. Hence k_{11} is induced by $St_3^1(i_7)$. In particular $k_{11} \circ j = St_3^1(i_7)$. Furthermore k_{11} generates $H^{11}(X_9;\mathbb{Z}_3)$.

It remains to calculate k_{12}. Consider $d_9 : E_9^{4,8} \to E_9^{13,0}$, for $a \in H^4(X_6;\mathbb{Z}_3)$

$$d_9(a \otimes \delta(i_4)) = a\delta(k_8) = a\delta St_3^1(i_4)$$

which is not zero for $a \neq 0$. So only elements that can contribute to $H^{12}(X_9)$ are: $\delta St_3^1(i_7)$ and $St_3^1\delta(i_7)$. In fact they do survive $d_{13} : E_{13}^{0,12} \to E_{13}^{13,0}$

$$d_{13}(\delta St_3^1(i_7)) = \delta St_3^1(k_8) = \delta d_{12}(St_3^1(i_7)) = 0$$

$$d_{13}(St_3^1\delta(i_7)) = St_3^1\delta(k_8) = St_3^1\delta St_3^1(i_4) = \delta St_3^2(i_4) = \delta(i_4^2) = 0$$

here we used $St_3^1\delta St_3^1 = \delta St_3^2 + St_3^2\delta$, and δ vanishes on integral classes. Hence $H^{12}(X_9;\mathbb{Z}_3)$ is generated by the two and the four dimensional classes of $H^{12}(X_6;\mathbb{Z}_3)$, and two new elements $\{\alpha,\beta\}$ with $j^*(\alpha) = \delta St_3^1(i_7)$ and $j^*(\beta) = St_3^1\delta(i_7)$. Since

$$j^*(\alpha) = \delta(j^* k_{11}) = j^*(\delta k_{11})$$

we have $\delta(k_{11}) = \alpha$. Now let $K(\mathbb{Z}_3, 10) \longrightarrow X_{10} \xrightarrow{\pi_9} X_9$ be the fibration pulled back from the universal fibration over $K(\mathbb{Z}_3, 11)$ by k_{11}. The next nonzero homotopy group $\pi_{11}BU(2)_{(3)} = \pi_{10}BU(2)_{(3)} = \mathbb{Z}_3$ gives the k invariant k_{12}. We claim that $H^{12}(X_{10}; \mathbb{Z}_3)$ is generated by the two and the four dimensional classes of $H^{12}(X_6; \mathbb{Z}_3)$, and a class $C = \pi_9^*(\beta)$. This implies $C = k_{12}$ and concludes the proof of the lemma.

We can check the claim from the spectral sequence of this fibration. i_{10} and $\delta(i_{10})$ transgresses to k_{11} and $\delta(k_{11}) = \alpha$, respectively. Hence, β is the only class that survives the spectral sequence besides classes coming from the base

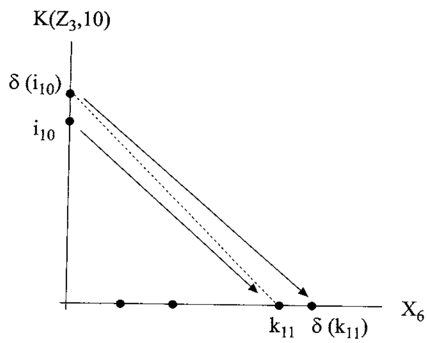

Figure 2.

□

The k_{11} and k_{12} gives a useful secondary cohomology operations. To compute these operation we need to know how they behave under pull-backs. Since $k_8 \circ f$ is null homotopic, it pulls back the trivial bundle. Let $\tilde{f}$ be the map covering f

$$\begin{array}{ccc} K(\mathbb{Z}_3, 7) \times BU(2) & \xrightarrow{\tilde{f}} & X_9 \\ \downarrow & & \downarrow \\ BU(2) & \xrightarrow{f} & X_6 \end{array}$$

Lemma 2.2 :

(a) $\tilde{f}^*(k_{11}) = St_3^1(i_7) \times 1 - i_7 \times c_2 + i_7 \times c_1^2$
(b) $\tilde{f}^*(k_{12}) = St_3^1\delta(i_7) \times 1$

Proof. By construction $\tilde{f}^*(k_{11})$ is nonzero since it has to restrict to $St_3^1(i_7)$ on the fiber. By considering f and $\tilde{f}$ as inclusions, we can apply Thomas' exact sequence of this relative fibration ([T], p.18) and get the exact sequence:

$$.. \longrightarrow H^{11}(X_9; \mathbb{Z}_3) \xrightarrow{\tilde{f}^*} H^{11}(K(\mathbb{Z}_3, 7) \times BU(2); \mathbb{Z}_3) \xrightarrow{\tau} H^{12}(X_6, BU(2); \mathbb{Z}_3)$$

where τ is the relative transgression. Since $H^{11}(BU(2)) = 0$ τ takes values in $H^{12}(X_6; \mathbb{Z}_3)$. If $g : K(\mathbb{Z}_3, 7) \times BU(2) \to X_6$ is the projection followed by f, then we can define a right action of $H^*(X_6;\mathbb{Z}_3)$ on $H^*(K(\mathbb{Z}_3,7)\times BU(2);\mathbb{Z}_3)$ by $u.v = u \smile g^*(v)$. By ([T] page 14), the transgression is equivariant under this action i.e. $\tau(u.v)=\tau(u)\smile v$. Hence

$$\tau(i_7\times c_2)=\tau((i_7\times 1)\smile(1\times c_2))=\tau((i_7\times 1)\smile g^*(i_4))=k_8 i_4$$

Similarly $\tau(i_7\times c_1^2)=k_8 i_2^2$. Since $\tau(St_3^1(i_7)\times 1)=St_3^1(k_8)=i_4k_8-i_2^2k_8$. So $St_3^1(i_7)\times 1-i_7\times c_2+i_7\times c_1^2$ lies in the kernel of τ, hence by the exact sequence this must be $\tilde{f}^*(k_{11})$ (recall k_{11} is a generator of $H^{11}(X_9;\mathbb{Z}_3)$).

We prove (b) similarly. In this case we need to look at the exact sequence

$$..\longrightarrow H^{12}(X_9;\mathbb{Z}_3)\xrightarrow{\tilde{f}^*} H^{12}(K(\mathbb{Z}_3,7)\times BU(2);\mathbb{Z}_3)\xrightarrow{\tau} H^{13}(X_6,BU(2);\mathbb{Z}_3)$$

(this sequence is exact up to dimension 13), and compute as before

$$\tau(St_3^1\delta(i_7)\times 1)=St_3^1\delta(k_8)\times 1=St_3^1\delta St_3^1(i_4)\times 1=0$$

□

3. On Algebraic Topology of Gauge Group

A quick way to obtain essential homology in $\mathcal{B}^*=\mathcal{B}^*(P)$ is the following: Assume $\pi_{k+3}SO(3)=\mathbb{Z}_p$, and consider the maps:

$$\begin{array}{ccc} \pi_{k+3}SO(3) & \xrightarrow{\varphi} & \pi_k(\mathcal{B}) \\ \tilde{\varphi}\searrow & & \swarrow h \\ & H_k(\mathcal{B}) & \end{array}$$

where h is the Hurewicz map, $\tilde{\varphi}=h\circ\varphi$, and φ is defined by sending an $SO(3)$-bundle $\alpha\to S^{k+4}$ to the $SO(3)$-bundle obtained by clutching α by P:

$$P\sharp\alpha\longrightarrow(S^k\times X)\sharp S^{k+4}=S^k\times X$$

Lemma 3.1:

(a) φ is a monomorphisim if every $SO(3)$ bundle
$\xi\longrightarrow S^{k+1}\times X-int(B^{k+5})$ with $\xi|1\times X=P$ extends to $S^{k+1}\times X$

(b) $\tilde{\varphi}$ is a monomorphisim if every $SO(3)$ bundle
$\xi\longrightarrow Z^{k+1}\times X-int(B^{k+5})$ with $\xi|1\times X=P$ extends to $Z^{k+1}\times X$
where Z is a $k+1$ dimensional compact oriented pseudo manifold (i.e. Z is a manifold in the complement of a codimension two subcomplex)

Proof. Proof of (a) is a special case of (b). To see (b): Suppose $\tilde{\varphi}(\alpha)=0$, this means that there is an integral cycle Z_0 with $\partial Z_0=S^k$, such that the bundle $P\sharp\alpha\longrightarrow S^k\times X$ extends to a bundle $\xi_0\longrightarrow Z_0\times X$. Let $Z=Z_0\cup_\partial D^{k+1}$. So there exists a bundle

$$\xi\longrightarrow Z^{k+1}\times X-int(B^{k+5})$$

which restricts to $\alpha \to S^{k+4}$ on the boundary. In particular ξ does not extend to $Z \times X$. (a) is proved similarly, except we use a disc D^{k+1} instead of Z^{k+1}.

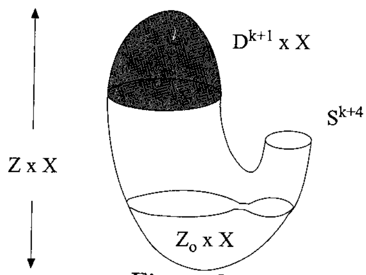

Figure 3.

Lemma 3.2 : If $\pi_1(\mathcal{B}^*(P)) = 0$, then

$$\pi_6(\mathcal{B}^*(P))_{(3)} = \begin{cases} Z_3 & \text{if } p_1(P) = 0 \mod 3 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Elements of $\pi_6(\mathcal{B}^*)_{(3)}$ are identified by the set of $SO(3)$ bundles on $D^6 \times X$ which are fixed on the boundary. As above, these bundles lie in

$$\mathcal{A}_6(X) = \pi_9 SO(3)_{(3)} = \mathbb{Z}_3$$

By Lemma 3.1 (a) $\pi_6(\mathcal{B}^*)_{(3)} = \mathbb{Z}_3$ provided every $SO(3)$ bundle

$$\xi \longrightarrow S^7 \times X - intB^{11}$$

with $\xi \mid 1 \times X = P$, extends to $S^7 \times X$; otherwise $\pi_6(\mathcal{B}^*)_{(3)} = 0$. Since $SO(3)$ bundles in question all lift to $U(2)$ bundles, it suffices to do this extension for $U(2)$ bundles. From Postnikov tower of $BU(2)$ we have:

$$\begin{array}{ccccccc} K(\mathbb{Z}_3,7) \times (S^7 \times X) & \xrightarrow{1\times g} & K(\mathbb{Z}_3,7) \times BU(2) & \xrightarrow{\tilde{f}} & X_9 & \xrightarrow{k_{11}} & K(\mathbb{Z}_3, 11) \\ s \uparrow\downarrow & & \downarrow & & \downarrow & & \\ S^7 \times X & \xrightarrow{g} & BU(2) & \xrightarrow{f} & X_6 & \xrightarrow{k_8} & K(\mathbb{Z}_2, 8) \end{array}$$

Here $f : BU(2) \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$ is given by (c_1, c_2), and g is the classifying map for P. Since $k_8 \circ f$ is null homotopic, f and $f \circ g$ induces trivial $K(\mathbb{Z}_3, 7)$-bundles over $BU(2)$ and $S^7 \times X$ respectively.

In particular $f \circ g$ lifts to X_9, and the number of different liftings are given by $H^7(S^7 \times X, 1 \times X; \mathbb{Z}_3) = H_4(X; \mathbb{Z}_3)$. Let $\tilde{f} : K(\mathbb{Z}_3, 7) \times BU(2) \to X_9$ be the map covering f. Then different liftings h of $f \circ g$ can be constructed by $h = \tilde{f} \circ (1 \times g) \circ s$, where $s = (\beta, id) : S^7 \times X \to K(\mathbb{Z}_3, 7) \times (S^7 \times X)$ is a section of the trivial bundle corresponding to a map $\beta : S^7 \times X \to K(\mathbb{Z}_3, 7)$. By Lemma 2.2 $\tilde{f}^*(k_{11}) = St_3^1(i_7) \times 1 - i_7 \times c_2 + i_7 \times c_1^2$, hence

$$h^*(k_{11}) = St_3^1(\beta) + \beta \smile (c_1^2(P) - c_2(P))$$

In general $\beta = \lambda\ [\ S^7 \times 1\ ]$, for some λ which gives:

$$h^*(k_{11}) = \lambda[S^7] \times (c_1^2(P) - c_2(P)) \mod 3$$

Since $p_1(P) = c_1^2(P) - 4c_2(P)$ we have $h^*(k_{11}) = 0$ if $p_1(P) = 0 \bmod 3$ □

Lemma 3.3 : If $\pi_1(\mathcal{B}^*(P)) = 0$, then

$$\pi_7(\mathcal{B}^*(P)) = \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \text{if } p_1(P) = 0 \bmod 3 \\ \mathbb{Z}_3 & \text{otherwise} \end{cases}$$

Furthermore Hurewicz map $h : \pi_7(\mathcal{B}^*)_{(3)} \to H_7(\mathcal{B}^*)$ is monomorphisim on the subgroup $\{0\} \oplus \mathbb{Z}_3$.

Proof. $\pi_7(\mathcal{B}^*)_{(3)}$ are the set of $SO(3)$ bundles on $D^7 \times X$ which are fixed on the boundary, which is a subset of

$$\mathcal{A}_7(X) = \pi_6 SO(3)_{(3)} \oplus \pi_{10} SO(3)_{(3)} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$$

The obstructions to realizing these bundles lie in $\mathcal{O}_7(X) = \pi_9 SO(3) = \mathbb{Z}_3$. This actually means that $\pi_7(\mathcal{B}^*)_{(3)}$ contains $\mathbb{Z}_3 \oplus \{0\}$, provided the corresponding obstruction in $\mathcal{O}_7(X)$ vanishes. This obstruction is the obstruction $k_{11}(\xi)$ to extending a bundle $\xi \longrightarrow S^7 \times X - int(B^{11})$ to $S^7 \times X$. As in Lemma 3.2

$$k_{11}(\xi) = [S^7] \times (c_1^2(P) - c_2(P)) \mod 3$$

So $\pi_7(\mathcal{B}^*)_{(3)}$ contains $\mathbb{Z}_3 \oplus \{0\}$ only if $p_1(P) = 0 \mod 3$. To see that hurewicz map is nonzero on $\{0\} \oplus \mathbb{Z}_3$ we use Lemma 3.1, i.e. we need to show that for any oriented pseudo manifold Z^8, every deformation of the $SO(3)$ bundle P

$$\Xi \longrightarrow Z^8 \times X - int(B^{12})$$

extends over $Z^8 \times X$. This is detected by the obstruction $k_{12}(\Xi)$. We claim that this obstruction vanishes. To see this pick $\beta : Z^8 \times X \to K(\mathbb{Z}_3, 7)$

$$\beta = a_7 \times 1 + a_5 \times \gamma_2 + a_3 \times [X]$$

with $a_i \in H^i(Z)$ and $\gamma_2 \in H^2(X)$; and evaluate $k_{12}(\Xi)$ as in Lemma 3.2 by using Lemma 2.2. We get:

$$k_{12}(\Xi) = St_3^1 \delta(a_3)$$

Now recall that Z in the Lemma 3.1 comes from a homology cycle $f : Z_0 \to \mathcal{B}^*$ bounding S^7. Since we assume $\pi_1(\mathcal{B}^*) = 0$, $H_2(\mathcal{B}^*; \mathbb{Z}) = \pi_2(\mathcal{B}^*)$. Hence by the Proposition of the introduction $H_2(\mathcal{B}^*; \mathbb{Z})$ has no odd torsion. We can assume that Z_0 is simply connected and $H_2(Z_0)$ has no odd torsion (collapse 1 and 2-spheres in Z_0, then extend f). Hence $H^3(Z;\mathbb{Z})$ has no 3 torsion , which implies $\delta(a_3) = 0$. □

References

[A] S. Akbulut, *Universal Relations in Gauge Theory* (to appear TAMS)

[AMR] S. Akbulut, T. Mrowka and Y. Ruan, *Torsion Classes and A Universal constraint on Donaldson invariants for odd manifolds,* TAMS, Vol. 347, No. 1, January (1995) 63-76.

[D] S. Donaldson, *Connections, cohomology and the intersection of four manifolds,* Jour. of Diff. Geometry, 24, (1986) 275-341

[DFN] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, *Modern Geometry-Mathods and Applications,* Springer-Verlag Grad. Texts in Math No.124 (1990)

[DK] S.K. Donaldson, and P. Kronheimer, *The Geometry of 4-manifolds,* Oxford Math. Mongraphs (1990).

[FS] R. Fintushel, and R. Stern, *2-torsion instanton invariants,* AMS Journal of Math. (1993).

[M] G. Masbaum, *On the cohomology of the classifying space of the gauge group over some 4-complexes,* Bull. Soc. France, 119, (1991), 1-31

[MT] R.E. Moser and M.C. Tangora, *Cohomology operations and applications in homotopy theory,* Harper and Row Publishers (1968)

[T] E. Thomas, *Seminar on fiber spaces,* Lecture notes in math. no. 13, Springer-Verlag (1966)

[To] H.Toda, *Composition methods in homotopy groups of spheres,* Annals of Math Studies, No.49

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