

ON UNIVERSAL RELATIONS IN GAUGE THEORY

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ABSTRACT. In this paper we study the algebraic topology of gauge group, and as a corollary we deduce some universal relations among Donaldson polynomials of smooth 4-manifolds.

1. INTRODUCTION

Let $P \rightarrow X^4$ be a principal $SU(2)$ or $SO(3)$ bundle over a smooth simply connected 4-manifold X . Hence in case P is an $SO(3)$ -bundle, it lifts to a $U(2)$ -bundle. It follows that when $w_2(X) = w_2(P)$ the parity of $c_2(P)$ is well defined number (i.e. it is independent of the lifting).

Recall that (e.g. [DK]) the space $\mathcal{B}^*(P)$ of gauge equivalence classes of irreducible connections in P is an infinite dimensional manifold with nontrivial topology. $\mathcal{B}^*(P)$ also admits a natural $SO(3)$ -bundle $\bar{\xi} \rightarrow \mathcal{B}^*(P)$ which is usually called the base point fibration. For a generic choice of a metric g on X , the gauge equivalence classes of anti-self-dual connections $\mathcal{M}(P, g) \subset \mathcal{B}^*(P)$ forms a smooth orientable (possibly noncompact) finite dimensional submanifold, and its bordism class in $\mathcal{B}^*(P)$ is independent of the metric g .

When the dimension $b_2^+(X)$ of the positive definite part of $H_2(X)$ is odd then the dimension of $\mathcal{M}(P, g)$ is an even integer $2d = 2d_k = 8k - 3(1 + b_2^+(X))$, where $k = -p_1(P)/4$. For $a_1, \dots, a_d \in H_2(X)$ Donaldson defines invariants $\Phi_d(a_1, \dots, a_d)$ by associating to each a_i a codimension two submanifold $V_i \subset \mathcal{M}(P, g)$ and counting the “intersection number”

$$\mathcal{M}(P, g) \cap V_1 \cap \dots \cap V_d.$$

These invariants are usually assembled into a single homogeneous polynomial: $\Phi_d = \Phi_{X,d} = \sum \Phi_k(a_1, \dots, a_d) a_1^* \dots a_d^*$ in the dual homology classes. There is yet another invariant $\Phi_d(a_1, \dots, a_{d-2}, 1)$ which is defined by the evaluation:

$$\langle p_1(\bar{\xi}), \mathcal{M}(P, g) \cap V_1 \cap \dots \cap V_{d-2} \rangle.$$

In this paper we compute some higher homotopy groups of $\mathcal{B}^*(P)$ generalizing the fundamental group calculation of [AMR]. We study the topology of $\mathcal{B}^*(P)$ and discuss its Postnikov tower. By relating the k invariants of the Postnikov tower of $\mathcal{B}^*(P)$ to the homotopy groups coming from the “ends” of $\mathcal{M}(P, g)$, we show that the various Donaldson polynomial invariants have to satisfy some universal relations. These calculations are carried by evaluating certain secondary cohomology operations associated to the Postnikov tower of the classifying space $BU(2)$,

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by the methods of Emery Thomas [T]. The use of k -invariants of $\mathcal{B}^*(P)$ brings a special importance to its homotopy groups, by enabling us to relate them to the polynomials Φ_d . The k -invariants are calculated by proving some divisibility conditions for certain “universal” cohomology classes of $\mathcal{B}^*(P)$. Our motivation for this divisibility theorem mainly comes from Masbaum’s results in [M].

The question of whether these polynomial invariants satisfy some universal relations was raised by Donaldson. Friedman and Morgan showed that in the case of complex surfaces with “large diffeomorphism group” the Donaldson invariants are polynomials in their first Chern class and the intersection form. In [R] Ruan showed that, under some conditions on P , a certain combination of the invariants Φ_d have to be even. Later this was generalized as

Theorem ([AMR]). *If $c_2(P)$ is odd and $w_2(P) = w_2(X)$, then for all classes $\alpha_i \in H^2(X)$ with $\alpha_i^2 \equiv 0 \pmod{2}$ the Donaldson invariants satisfy:*

$$\sum_{i < j} q(a_i, a_j) \Phi_d(a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{d+2}) = 0 \pmod{2}.$$

where $q(a_i, a_j)$ denotes the intersection number of a_i with a_j .

In [FS] Fintushel and Stern proved some mod 2 vanishing results for the invariants Φ_d . Our main result gives a way of relating different degree polynomial invariants Φ_d and Φ_{d+4} to each other:

Theorem 5.4. *If $P \rightarrow X$ is an $SO(3)$ -bundle over a closed smooth simply connected 4-manifold, assume that $c_2(P)$ is even when $w_2(P) = w_2(X)$. Then for any basis b_1, \dots, b_n of $H_2(X)$ and $a_1, \dots, a_{d+2} \in H_2(X)$, we have*

$$\begin{aligned} & \sum_{i,j} \bar{q}(b_i, b_j) \Phi_{d+4}(a_1, \dots, a_{d+2}, b_i, b_j) + (d+1) \Phi_{d+4}(a_1, \dots, a_{d+2}, 1) \\ & \equiv \pm d(d-1) \sum_{i < j} q(a_i, a_j) \Phi_d(a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{d+2}) \pmod{3} \end{aligned}$$

where $(\bar{q}(b_i, b_j))$ denotes the inverse of the intersection matrix $(q(b_i, b_j))$.

The sign ambiguity in the above identity can be decided by evaluating it on an example, which we did not do in this paper. We also prove some mod p congruences between Donaldson polynomials. Some of the calculations of this paper are generalized in [A]. We hope to explore further generalizations in a future paper. If one views Donaldson invariants as a kind of nonlinear cohomology, then these relations may be thought of as “cohomology operations” of this theory. At the end we apply these results to the complex surfaces with large diffeomorphism group to get more concrete conclusions. We would like to thank Y. Ruan for his many motivating questions and enthusiastic support of this research. Finally I would like to thank John Havlicek for reading this paper and making valuable suggestions.

2. TOPOLOGY OF $\mathcal{B}^*(P)$

Let X be a closed smooth oriented 4-manifold with $\pi_1(X) = 0$, and $P \rightarrow X$ be a principal G -bundle, where $G = SO(3)$, or $SU(2)$. Let $C(G)$ denote the center of G . Let $\mathcal{A}(P)$ be the space of connections on P , and let $\mathcal{A}^*(P)$ denote the subspace of the irreducible connections on P . Let $\mathcal{G}(P)$ be the gauge group acting on $\mathcal{A}(P)$.

The quotient $\hat{\mathcal{G}}(P) = \mathcal{G}(P)/C(G)$ acts on $\mathcal{A}^*(P)$ freely giving us the principal fibration:

$$\hat{\mathcal{G}}(P) \longrightarrow \mathcal{A}^*(P) \longrightarrow \mathcal{B}^*(P)$$

where $\mathcal{B}^*(P) = \mathcal{A}^*(P)/\hat{\mathcal{G}}(P)$. For brevity the dependence of these spaces on P sometimes will be dropped from their notation. Since the total space of the fibration is contractible, we can identify $\mathcal{B}^*(P) = B_{\hat{\mathcal{G}}}$. Recall that the following proposition allows us to compute the homotopy groups of $\mathcal{B}^*(P)$ (cf. [AMR]).

Proposition 2.1. *There is a weak homotopy equivalence:*

$$\mathcal{B}^*(P) \simeq \text{Map}^P(X, BSO(3))$$

where the right-hand side denotes the component of the space of (unbased) maps inducing the bundle P .

The evaluation map $\text{ev} : X \times \text{Map}^P(X, BSO(3)) \longrightarrow BSO(3)$ induces an $SO(3)$ -bundle $\xi \rightarrow X \times \mathcal{B}^*$. Now let $\mu : H_2(X) \rightarrow H^2(\mathcal{B}^*)$ be the usual μ -map, defined by the homology slant product $\mu(a) = -p_1(\xi)/a$. To be precise Donaldson's μ map which we will call μ^* is defined to be $\mu/4$. In order to make μ^* an integral class we have to define it on a smaller subgroup:

$$\mu^* : A_2(X) = \{a \in H_2(X) \mid \langle w_2(P), a \rangle = 0\} \rightarrow H^*(\mathcal{B}^*; \mathbf{Z}).$$

$2\mu^*$ is integral on $H_2(X)$. It is by this map μ^* which the Donaldson's invariants are defined (e.g. [DK]).

For a basis a_1, a_2, \dots, a_n of $H_2(X)$ we define

$$\Omega = \frac{1}{4} \sum_{i,j} \bar{q}(a_i, a_j) \mu(a_i) \mu(a_j) \in H^4(\mathcal{B}^*; \mathbf{Z}).$$

Here $(\bar{q}(a_i, a_j))$ denotes the inverse of the intersection matrix $(q(a_i, a_j))$. It is easy to check that Ω is independent of the chosen basis, and is a **natural class** in the sense of [M]. This means that if we write $\Omega = \Omega(X)$ then for any degree one map $f : X' \rightarrow X$ the induced map $F : \text{Map}^P(X, BSO(3)) \rightarrow \text{Map}^{P'}(X', BSO(3))$, where $P' = f^*(P)$, has the property: $F^*(\Omega(X')) = \Omega(X)$.

By restriction, the bundle ξ gives a bundle $\bar{\xi} \rightarrow \mathcal{B}^*$ which is usually called the base point fibration. Let $\bar{p}_1 \in H^4(\mathcal{B}^*; \mathbf{Z})$ denote the first Pontryagin class of $\bar{\xi}$, i.e. $\bar{p}_1 = -\mu(1)$. Let p_1 denote the first Pontryagin number of P , i.e. $p_1 = \langle p_1(P), [X] \rangle$ where $[X]$ is the generator of $H_4(X)$. The following theorem is essentially a certain generalization of a result of Masbaum [M].

Proposition 2.2. *If X is even then $\Omega - p_1 \bar{p}_1 \in H^4(\mathcal{B}^*; \mathbf{Z})$ is divisible by 6. Furthermore, if $\pi_1(\mathcal{B}^*) = 0$ then $\Omega - p_1 \bar{p}_1$ is divisible by 12.*

Proof. Let $\bar{P} \rightarrow X$ be a lifting of P to a $U(2)$ -bundle. Principal fibrations:

$$S^1 \rightarrow U(2) \rightarrow SO(3) \rightarrow BS^1 \rightarrow BU(2) \longrightarrow BSO(3)$$

induce a principal fibration (action is the pointwise multiplication of maps):

$$\text{Map}^0(X, K(\mathbf{Z}, 2)) \rightarrow \text{Map}^{\bar{P}}(X, BU(2)) \xrightarrow{\pi} \text{Map}^P(X, BSO(3))$$

where $\text{Map}^0(X, K(\mathbf{Z}, 2))$ means the component of the maps inducing 0 in the second cohomology (since $BS^1 = K(\mathbf{Z}, 2)$). The fiber of this fibration is $K(\mathbf{Z}, 2)$. So, we get a fibration $K(\mathbf{Z}, 2) \rightarrow \tilde{\mathcal{B}} \xrightarrow{\pi} \mathcal{B}^*$, where $\tilde{\mathcal{B}} = \text{Map}^{\bar{P}}(X, BU(2))$.

As above we can consider the $U(2)$ -bundle $\eta \rightarrow X \times \tilde{\mathcal{B}}$ induced by the evaluation map $X \times \tilde{\mathcal{B}} \rightarrow BU(2)$, and define the corresponding map $\tilde{\mu} : H_2(X) \rightarrow H^2(\tilde{\mathcal{B}})$, by taking the slant product with $-p_1(\eta)$. Let

$$\tilde{\Omega} = \frac{1}{4} \sum_{i,j} \bar{q}(a_i, a_j) \tilde{\mu}(a_i) \tilde{\mu}(a_j) \in H^4(\tilde{\mathcal{B}}; \mathbf{Z}).$$

Similarly by restricting η we get a base point fibration $\bar{\eta} \rightarrow \tilde{\mathcal{B}}$. Since $p_1(\eta) = p_1(\pi^*\xi)$, we have $\pi^*(\Omega - p_1 p_1(\xi)) = \tilde{\Omega} - p_1 p_1(\bar{\eta})$. We claim $\pi^* : H^4(\mathcal{B}^*) \rightarrow H^4(\tilde{\mathcal{B}})$ is split injective when $\pi_1(\mathcal{B}^*) = 0$, and may be multiplication by 2 in general. By this claim, Proposition 2.2 follows if we show $\tilde{\Omega} - p_1 p_1(\bar{\eta})$ is divisible by 12.

Let us first prove the claim: First of all the fibration $K(\mathbf{Z}, 2) \rightarrow \tilde{\mathcal{B}} \xrightarrow{\pi} \mathcal{B}^*$ is induced from $K(\mathbf{Z}, 2) \rightarrow BU(2) \rightarrow BSO(3)$ by the evaluation map, it has a constant coefficient system. By the spectral sequence of the fibration $K(\mathbf{Z}, 2) \rightarrow \tilde{\mathcal{B}} \xrightarrow{\pi} \mathcal{B}^*$, π^* is given by the following composition:

$$\pi^* : H^4(\mathcal{B}^*) = E_2^{4,0} = E_\infty^{4,0} = F^{4,0} = F^{3,1} \subset F^{2,2} = F^{1,3} \subset F^{0,4} = H^4(\tilde{\mathcal{B}})$$

where $E_\infty^{p,q} = F^{p,q}/F^{p+1,q-1}$. The following is the E_2 of the spectral sequence:

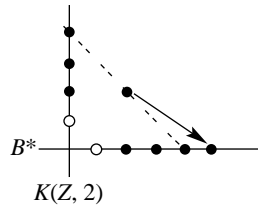


FIGURE 1

By Proposition 4.1 $\pi_1(\mathcal{B}^*) = 0$ or \mathbf{Z}_2 , $E_\infty^{0,4}$ is free and $E_\infty^{2,2}$ is free up to a \mathbf{Z}_2 summand. Then, by the exactness of $0 \rightarrow E_\infty^{2,2} \rightarrow H^4(\tilde{\mathcal{B}})/H^4(\mathcal{B}^*) \rightarrow E_\infty^{0,4} \rightarrow 0$ the group $H^4(\tilde{\mathcal{B}})/H^4(\mathcal{B}^*)$ is free up to a \mathbf{Z}_2 summand or free when $\pi_1(\mathcal{B}^*) = \mathbf{Z}_2$. Hence the proof of the claim follows from the exact sequence

$$0 \rightarrow H^4(\mathcal{B}^*) \xrightarrow{\pi^*} H^4(\tilde{\mathcal{B}}) \rightarrow H^4(\tilde{\mathcal{B}})/H^4(\mathcal{B}^*) \rightarrow 0$$

Now we have to show that $\tilde{\Omega} - p_1 \bar{p}_1$ is divisible by 12. A similar fact was proved in [M] in the case of $BSU(2)$. We will modify [M] to fit our $BU(2)$ case.

By naturality, it suffices to prove this for $X = \sharp_g S^2 \times S^2$ [M]. Consider the commutative diagram obtained by stabilizing the evaluation maps:

$$(*) \quad \begin{array}{ccc} X \times \text{Map}^{\bar{P}}(X, BU(2)) & \xrightarrow{\eta} & BU(2) \\ \downarrow 1 \times j & & \downarrow j \\ X \times \text{Map}^{\bar{P}}(X, BU) & \xrightarrow{\tilde{\eta}} & BU \end{array}$$

Since $BU \simeq BSU \times K(\mathbf{Z}, 2)$, by calling $k = c_2(\bar{P})$ we get

$$\begin{aligned} \text{Map}^{\bar{P}}(X, BU) &\simeq \text{Map}^k(X, BSU) \times \text{Map}(X, K(\mathbf{Z}, 2)) \\ &\simeq \text{Map}^k(X, BSU) \times K(\mathbf{Z}, 2) \end{aligned}$$

By [M] there is a homotopy equivalence:

$$\phi : \text{Map}^k(X, BSU) \xrightarrow{\simeq} BSU \times (BU)^{2g} \times BU$$

given by $\phi(f) = (f(pt), P(f), Q(f))$ for some P and Q . The evaluation map $\eta_0 : X \times \text{Map}^k(X, BSU) \rightarrow BSU$ corresponds to

$$(x, z, (y_1, y'_1, \dots, y_g, y'_g), y) \longrightarrow z \cdot \prod_{i=1}^g F(r_i(x), y_i) \cdot F(r'_i(x), y'_i) \cdot \tilde{F}([x], y)$$

where $F : S^2 \times BU \rightarrow BSU$, $\tilde{F} : S^4 \times BU \rightarrow BSU$ are adjoints of the Bott equivalences $BU \simeq \Omega^2 BSU$, $BU \simeq \Omega_k^4 BSU$, and $r_i, r'_i : X \rightarrow S^2$ are the maps obtained by first collapsing $X = \#_g S^2 \times S^2$ to the i th factor $S^2 \times S^2$ and then projecting to the first and the second S^2 factors respectively. Also $[x]$ denotes the image of $x \in X$ in $X/\bigvee_{i=1}^g S^2 \vee S^2 \approx S^4$.

If c denotes the total Chern class and s_n denotes the Newton polynomials, and $\sigma_i \in H^i(S^i; \mathbf{Z})$, $i = 2, 4$, are the standard generators then a routine calculation (cf. [M]) gives

$$(1) \quad F^*(c) = 1 + \sigma_2 \times A, \quad \text{with } A = \sum_{n \geq 1} (-1)^{n+1} s_n(c_1, c_2, \dots),$$

$$(2) \quad \tilde{F}^*(c) = 1 + \sigma_4 \times B, \quad \text{with } B = k + \sum_{n \geq 1} (-1)^{n+1} (n+1) s_n(c_1, c_2, \dots),$$

Notice in the splitting $BU \simeq BSU \times K(\mathbf{Z}, 2)$ the universal bundle ω corresponds to $\omega_0 \oplus L$, where $\omega_0 = \omega \oplus L^{-1}$, and $L = \det(\omega)$. Hence the bundle induced by $\tilde{\eta}$ splits as: $\tilde{\eta} = \eta_0 \oplus L$. So the total Chern class is $c(\tilde{\eta}) = c(L)c(\eta_0)$.

Let $\{a_1, a'_1, \dots, a_g, a'_g\}$ be the standard basis of $H^2(X)$, and for the sake of simplicity let $[X]$ also denote the generator of $H^4(X)$. Since the multiplication on BSU is induced by the Whitney sum of bundles the total Chern class of $\tilde{\eta}$ can be written as

$$c(\tilde{\eta}) = c(L)(1 \times c) \prod_{i=1}^g (1 + a_i \times A_i)(1 + a'_i \times A'_i)(1 + [X] \times B)$$

$$= c(L)[1 \times c + \sum a_i \times cA_i + \sum a'_i \times cA'_i + [X] \times c(B + \sum A_i A'_i)].$$

Here classes c , A_i , A'_i , and B in $H^*(\text{Map}^k(X, BSU))$ correspond in the obvious way to the components of $\text{Map}^k(X, BSU) \simeq BSU \times (BU)^{2g} \times BU$. Here we also used $a_i a'_i = \delta_{ij} [X]$ and $a_i a_j = 0 = a'_i a'_j$.

Now write: $c(L) = \alpha \times 1 + 1 \times \tilde{\beta}$, where $\alpha = \sum \lambda_i a_i + \sum \lambda'_i a'_i$ and $\tilde{\beta} = 1 + \beta$. By calling $\tilde{c} = \tilde{\beta}c$ we get

$$c(\tilde{\eta}) = 1 \times \tilde{c} + \sum a_i \times \tilde{c}A_i + \sum a'_i \times \tilde{c}A'_i + [X] \times \tilde{c}(B + \sum A_i A'_i)$$

$$+ \sum [X] \times c\lambda'_i A_i + \sum [X] \times c\lambda_i A'_i + \alpha \times c$$

$$= 1 \times \tilde{c} + \sum a_i \times c(\tilde{\beta}A_i + \lambda_i) + \sum a'_i \times c(\tilde{\beta}A'_i + \lambda'_i)$$

$$+ [X] \times c(\tilde{\beta}B + \tilde{\beta} \sum A_i A'_i + \sum \lambda'_i A_i + \sum \lambda_i A'_i).$$

Now express the total Chern class of the bundle $\eta \longrightarrow X \times \text{Map}^{\bar{P}}(X, BU(2))$

$$(3) \quad c(\eta) = 1 \times \bar{c} + \sum a_i \times b_i + \sum a'_i \times b'_i + c(\bar{P}) \times 1$$

where $c(\bar{P}) = c_1(\bar{P}) + c_2(\bar{P})$. By expressing in the basis $c_1(\bar{P}) = \sum \mu_i a_i + \sum \mu'_i a'_i$ and $c_2(\bar{P}) = k[X]$ we can write

$$c(\eta) = 1 \times \bar{c} + \sum a_i \times (b_i + \mu_i) + \sum a'_i \times (b'_i + \mu'_i) + k[X] \times 1.$$

From commutativity of the diagram (*) we have $J^*(c(\tilde{\eta})) = c(\eta)$, where $J = 1 \times j$. Hence $j^*(\bar{c}) = j^*(\tilde{\beta}c) = \bar{c}$ and

$$\begin{aligned} j^*(c(\tilde{\beta}A_i + \lambda_i)) &= b_i + \mu_i, \\ j^*(c(\tilde{\beta}A'_i + \lambda'_i)) &= b'_i + \mu'_i, \\ j^*(c(\tilde{\beta}B + \tilde{\beta} \sum A_i A'_i + \sum \lambda'_i A_i + \sum \lambda_i A'_i)) &= k. \end{aligned}$$

Call $j^*(\tilde{\beta}) = \bar{\beta}$. We can write $j^*(c) = 1 + q$, where q is a 4 dimensional cohomology class. So $\bar{\beta} = 1 + \bar{c}_1$ and $q = \bar{c}_2$, where $\bar{c} = 1 + \bar{c}_1 + \dots$. From the above identities we get $\mu_i = \lambda_i$ and $\mu'_i = \lambda'_i$, and

$$\begin{aligned} \bar{\beta}j^*(A_i) &= (b_i + \lambda_i)/(1 + q) - \lambda_i, \\ \bar{\beta}j^*(A'_i) &= (b'_i + \lambda'_i)/(1 + q) - \lambda'_i, \\ j^*(\bar{\beta}^2 B + \sum \tilde{\beta}A_i \tilde{\beta}A'_i + \sum \lambda'_i \tilde{\beta}A_i + \sum \lambda_i \tilde{\beta}A'_i) &= k\bar{\beta}/(1 + q). \end{aligned}$$

By substituting the first two equalities in the third we see

$$\begin{aligned} \bar{\beta}^2 j^*(B) &= k\bar{\beta}/(1 + q) - \sum [(b_i + \lambda_i)/(1 + q) - \lambda_i] [(b'_i + \lambda'_i)/(1 + q) - \lambda'_i] \\ &\quad - \sum \lambda'_i [(b_i + \lambda_i)/(1 + q) - \lambda_i] - \sum \lambda_i [(b'_i + \lambda'_i)/(1 + q) - \lambda'_i] \\ &= k\bar{\beta}/(1 + q) - 1/(1 + q)^2 \sum (b_i + \lambda_i)(b'_i + \lambda'_i) + \sum \lambda_i \lambda'_i \end{aligned}$$

$$(4) \quad (1 + q)^2 \bar{\beta}^2 j^*(B) = (1 + q)k\bar{\beta} - \sum (b_i + \lambda_i)(b'_i + \lambda'_i) + (1 + q)^2 \sum \lambda_i \lambda'_i.$$

By the expression of B above (2), we have

$$B = k + \sum_{n \geq 1} (-1)^{n+1} (n + 1) s_n(c_1(\mathcal{Z}), c_2(\mathcal{Z}), \dots)$$

where $\mathcal{Z} \rightarrow \tilde{\mathcal{B}}$ is the bundle induced by composing the projection π with Q :

$$\text{Map}^{\bar{P}}(X, BU) \xrightarrow{\pi} \text{Map}^k(X, BSU) \xrightarrow{Q} BU.$$

We do not need to identify \mathcal{Z} , but according to [M] Q represents the K -theory slant product bundle $\eta_0/[X]$. Call $j^*(\mathcal{Z}) = \zeta$. By expanding both sides, and equating the two dimensional cohomology classes in the last equality (4) we get

$$(5) \quad \sum (b_i \lambda'_i + b'_i \lambda_i) = -k\bar{c}_1 - 2c_1(\zeta).$$

By equating the four dimensional classes of both sides, and recalling that $q = \bar{c}_2$, we get $k\bar{c}_1^2 + 2k\bar{c}_2 + 4\bar{c}_1 c_1(\zeta) - 3c_1(\zeta)^2 + 6c_2(\zeta) = k\bar{c}_2 - \sum b_i b'_i + 2\bar{c}_2 \sum \lambda_i \lambda'_i$. Call $e = \langle c_1(\bar{P})^2, [X] \rangle$. Recall that $k = \langle c_2(\bar{P}), [X] \rangle$, and $e = 2 \sum \lambda_i \lambda'_i$. From the last equality

$$(6) \quad \sum b_i b'_i = (e - k)\bar{c}_2 - k\bar{c}_1^2 - 4\bar{c}_1 c_1(\zeta) + 3c_1(\zeta)^2 - 6c_2(\zeta).$$

Alternatively, we can derive (5) and (6) by applying Atiyah-Singer's index theorem for families [AS] to $\eta \rightarrow X \times \text{Map}^{\bar{P}}(X, BU(2))$. In our case since the Todd genus $\text{td}(X) = 1$ the families of index theorem gives

$$\text{ch}(\zeta) = \text{ch}(j^*(\eta_0/[X])) = \text{ch}(j^*(\tilde{\eta} \oplus L^{-1})/[X]) = \text{ch}(\eta)/[X] - \text{ch}(j^*L)/[X].$$

By plugging the values of the right-hand side from $c(j^*L) = c_1(\bar{P}) \times 1 + 1 \times \bar{c}_1$ and from (3) gives also the identities (5) and (6).

The second Chern and the first Pontryagin classes of any $U(2)$ -bundle are related by $p_1 = c_1^2 - 4c_2$, and $c(\eta) = 1 + c_1(\eta) + c_2(\eta) + \dots$. Hence

$$p_1(\eta) = 1 \times \bar{p}_1 + p_1(\bar{P}) \times 1 + \sum a_i \times 2(\lambda_i \bar{c}_1 - 2b_i) + \sum a'_i \times 2(\lambda'_i \bar{c}_1 - 2b'_i),$$

$$\tilde{\Omega} = 2 \sum (2b'_i - \lambda'_i \bar{c}_1)(2b_i - \lambda_i \bar{c}_1).$$

By expanding Ω and substituting from (5) and (6) above we get

$$\tilde{\Omega} = 8(e - k)\bar{c}_2 + (e - 4k)\bar{c}_1^2 - 24 [\bar{c}_1 c_1(\zeta) - c_1(\zeta)^2 + 2c_2(\zeta)].$$

By substituting $\bar{c}_1^2 = \bar{p}_1 + 4\bar{c}_2$ and $e = p_1 + 4k$ we get

$$\tilde{\Omega} = p_1 \bar{p}_1 + 12(p_1 + 2k)\bar{c}_2 - 24 [\bar{c}_1 c_1(\zeta) - c_1(\zeta)^2 + 2c_2(\zeta)]. \quad \square$$

We can also prove a version of Proposition 2.2 when X is odd: We can find a degree 2 map from an even manifold $F : Y \rightarrow X$, and by naturality properties of Ω we get $F^*(\Omega(Y)) = 4\Omega(X)$.

Corollary 2.3. *If X is odd, then $4\Omega - p_1 \bar{p}_1 = 0 \pmod{6}$. If furthermore $\pi_1(\mathcal{B}^*) = 0$ then $4\Omega - p_1 \bar{p}_1 = 0 \pmod{12}$, in particular, if X is odd then $p_1 \bar{p}_1 = 0 \pmod{4}$.*

Since $\pi_1(\mathcal{B}^*)$ is 0 or \mathbf{Z}_2 , torsion subgroup of $H^2(\mathcal{B}^*)$ is 0 or \mathbf{Z}_2 . Formula (5) of the above proof gives a result of [D]; namely if $P \rightarrow X$ is an $SU(2)$ -bundle (i.e. λ_i and $\lambda'_i = 0$) with $c_2(P) = k$ odd, then the universal fibration $\xi \rightarrow X \times \mathcal{B}^*$ lifts to a $SU(2)$ -bundle (i.e. \bar{c}_1 is even). Also notice that $c_2(P)$ odd implies $\pi_1(\mathcal{B}^*) = 0$ (Proposition 4.1).

In case of $SU(2)$ -bundles the fibration $K(\mathbf{Z}_2, 1) \rightarrow BSU(2) \rightarrow BSO(3)$ induces the fibration

$$K(\mathbf{Z}_2, 1) \rightarrow \text{Map}(X, BSU(2)) \xrightarrow{\pi} \text{Map}(X, BSO(3)).$$

By [M] $\pi^*(p_1 \bar{p}_1 + 2m\Omega) \bar{p}_1^{m-1} \in H^{4m}(\text{Map}(X, BSU(2)))$ is divisible by $2m + 1$. π induces isomorphism in cohomology groups with \mathbf{Z}_{2m+1} coefficients hence:

Proposition 2.4. *If $P \rightarrow X$ is an $SU(2)$ -bundle, then the class $(\Omega - p_1 \bar{p}_1) \bar{p}_1^{m-1}$ in $H^{4m}(\mathcal{B}^*)$ is divisible by $2m + 1$.*

3. POSTNIKOV TOWER OF $BU(2)$ AND THE ASSOCIATED COHOMOLOGY OPERATIONS

Computation of the homotopy groups of $\mathcal{B}^*(P)$ can be reduced to the problem counting number of $U(2)$ -bundles and extending $U(2)$ -bundles on associated spaces. Postnikov tower of $BU(2)$ is most suited for this type of problem. It consists of maps and spaces $BU(2) \rightarrow \dots \rightarrow X_{n+1} \rightarrow X_n \rightarrow \dots \rightarrow X_1$ and cohomology classes $k_{n+2} \in H^{n+2}(X_n; \pi_{n+1})$, where $\pi_n = \pi_n BU(2)$. Existence of an $U(2)$ -bundle over the $(n + 1)$ -skeleton of a space Z is equivalent to having a map $f_n : Z \rightarrow X_n$, and extending this bundle to an $(n + 2)$ -skeleton (after readjusting over the $(n + 1)$ -cells) is equivalent to showing $f_n^*(k_{n+2}) = 0$. Each $X_{n+2} \rightarrow X_n$ is a fibration with fiber $K(\pi_{n+1}, n + 1)$, obtained by pulling back the universal fibration by the map $k_{n+2} : X_n \rightarrow K(\pi_{n+1}, n + 2)$. In particular the fundamental cohomology class of the fiber transgresses to k_{n+2} . At the end of this section we will compute some useful secondary cohomology operations induced by the Postnikov tower of $BU(2)$.

Proposition 3.1. *The 6 stage Postnikov tower of $BU(2)$ is given by :*

$$\begin{array}{ccc} K(\mathbf{Z}_2, 6) & \xrightarrow{j} X_6 & \xrightarrow{k_8^1 \times k_8^2} K(\mathbf{Z}_4, 8) \times K(\mathbf{Z}_3, 8) \\ & \pi_5 \downarrow & \\ K(\mathbf{Z}_2, 5) & \xrightarrow{j} X_5 & \xrightarrow{k_7} K(\mathbf{Z}_2, 7) \\ & \pi_4 \downarrow & \\ BU(2) & \xrightarrow{f} X_4 & \xrightarrow{k_6} K(\mathbf{Z}_2, 6) \end{array}$$

where $X_4 = K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 4)$, and $f = (c_1, c_2)$ is given by the Chern classes. Also if i_n denotes the corresponding fundamental classes of $H^n(K(\mathbf{Z}, n) : \mathbf{Z}_2)$ and $H^n(K(\mathbf{Z}_2, n) : \mathbf{Z}_2)$, and Sq^k, St_3^k are the k -Steenrod square and cube operations, and $\rho_2 : K(\mathbf{Z}_4, 8) \rightarrow K(\mathbf{Z}_2, 8)$ is the mod 2 reduction then:

$$\begin{aligned} k_6 &= Sq^2 i_4 + i_2 i_4, \\ j \circ k_7 &= Sq^2(i_5), \\ \rho_2 \circ k_8^1 &= r \circ \pi_5, \text{ where } r : X_5 \rightarrow K(\mathbf{Z}_2, 8) \text{ with } r \circ j = Sq^2 Sq^1(i_5), \\ k_8^2 &= s \circ \pi_4 \circ \pi_5, \text{ where } s : X_4 \rightarrow K(\mathbf{Z}_3, 8) \text{ with } s = St_3^1(i_4) - i_4 i_2^2 - i_4^2. \end{aligned}$$

Proof. The 4-skeleton of $BU(2)$ is $K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 4)$ and the map $f = (c_1, c_2) : BU(2) \rightarrow K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 4)$ induces isomorphism on homotopy up to and including dimension 4. If F_5 is the fiber of the map f , then $\pi_k(F_5) = 0$ for all $k \leq 4$ and $\pi_5(F_5) = \pi_5(BSU(2)) = \mathbf{Z}_2$. By the Serre exact sequence the first k -invariant k_6 is the image of the fundamental class under transgression τ

$$\dots \longrightarrow H^5(F_5; \mathbf{Z}_2) \xrightarrow{\tau} H^6(X_4; \mathbf{Z}_2) \xrightarrow{f^*} H^6(BU(2); \mathbf{Z}_2).$$

So k_6 is the unique element in $H^6(X_4; \mathbf{Z}_2)$ with $f^*(k_6) = 0$. We claim that $k_6 = Sq^2 i_4 + i_2 i_4$. To check this we need to show that $f^*(k_6) = Sq^2 c_2 + c_1 c_2$ vanishes. By the splitting principle we can assume that the universal bundle is a direct sum of line bundles $L_1 \oplus L_2$ hence

$$\begin{aligned} c_1 &= c_1 L_1 + c_1 L_2, & c_2 &= c_1 L_1 \cdot c_1 L_2, \\ Sq^2 c_2 + c_1 c_2 &= 2((c_1 L_1)^2 \cdot c_1 L_2 + c_1 L_1 \cdot (c_1 L_2)^2) = 0 \pmod{2}. \end{aligned}$$

Let $K(\mathbf{Z}_2, 5) \xrightarrow{j} X_5 \xrightarrow{\pi_4} X_4$ be the induced fibration by k_6 from the universal fibration over $K(\mathbf{Z}_2, 6)$. By construction the fundamental class of $K(\mathbf{Z}_2, 5)$ transgresses to k_6 , and f lifts to a map $f_5 : BU(2) \rightarrow X_5$. The fiber F_6 of this map f_5 is 5-connected, and $\pi_6(F_6) = \pi_6 BU(2) = \mathbf{Z}_2$. The transgression of the fundamental class of $H^6(F_5; \mathbf{Z}_2)$ is the next k -invariant $k_7 \in H^7(X_5; \mathbf{Z}_2)$. This class has the property $f_5^*(k_7) = 0$ in fact by the Serre exact sequence

$$\dots \longrightarrow H^6(F_6; \mathbf{Z}_2) \xrightarrow{\tau} H^7(X_5; \mathbf{Z}_2) \xrightarrow{f_5^*} H^7(BU(2); \mathbf{Z}_2) = 0.$$

By using the spectral sequence of the fibration $K(\mathbf{Z}_2, 5) \rightarrow X_5 \rightarrow X_4$ we see that k_7 is the class as described in the statement of the theorem:

$$E_2^{p,q} = H^p(X_4; \mathbf{Z}_2) \otimes H^q(K(\mathbf{Z}_2, 5); \mathbf{Z}_2).$$

Let us denote the two and four dimensional fundamental classes of X_4 by a and b , respectively. Since these come from integral classes $Sq^1(a) = Sq^1(b) = 0$. By the spectral sequence the only classes that can possibly contribute to the seven dimensional cohomology of X_5 are: $Sq^2(i_5)$, $a \otimes i_5$, and the seven dimensional class $Sq^3(b)$ of X_4 . By the transgression relation $d_6(i_5) = k_6 = Sq^2(b) + ab$, we have

$$d_7(Sq^1(i_5)) = Sq^1(k_6) = Sq^1 Sq^2(b) = Sq^3(b)$$

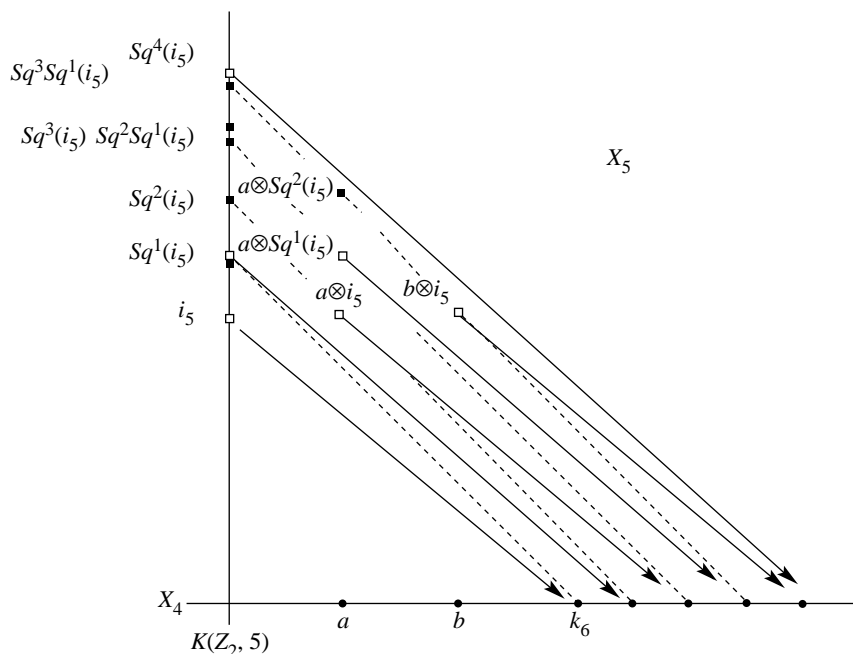


FIGURE 2

also

$$d_6(a \otimes i_5) = ak_6 = aSq^2(b) + a^2b \neq 0$$

Hence $Sq^2(i_5)$ is the only possible class that could contribute to the homology, and in fact this is the case since

$$\begin{aligned} d_8(Sq^2(i_5)) &= Sq^2(k_6) = Sq^2(Sq^2(b) + ab) = Sq^2Sq^2(b) + a^2b + aSq^2(b) \\ &= Sq^3Sq^1(b) + a^2b + aSq^2(b) = a^2b + aSq^2(b) \end{aligned}$$

and the last element has already been killed by $d_6(a \otimes i_5)$. So, $Sq^2(k_6)$ survives to give the only homology class in $H^7(X_5; \mathbf{Z}_2)$ and this must be the class k_7 by the above exact sequence. From this spectral sequence, by a routine calculation, we also get that $H^8(X_5; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and is generated by $\{a^2b, a^4, b^2, \alpha, \beta\}$ with the properties: $j^*(\alpha) = Sq^3(i_5)$, and $j^*(\beta) = Sq^2Sq^1(i_5)$. Furthermore $H^9(X_5; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and is generated by $\{\gamma, ak_7\}$, such that $j^*(\gamma) = Sq^3Sq^1(i_5)$. In Figure 2 we indicate the nonzero differentials in the spectral sequence.

Finally, we iterate this process for the last time by taking the fibration $K(\mathbf{Z}_2, 6) \rightarrow X_6 \xrightarrow{\pi_5} X_5$ induced from $K(\mathbf{Z}_2, 7)$ by k_7 . Again, the fundamental class of $K(\mathbf{Z}_2, 6)$ transgresses to k_7 , and f_5 lifts to $f_6 : BU(2) \rightarrow X_6$. The 6-connected fiber F_7 of f_6 has the first homotopy $\pi_7(F_7) = \pi_7BU(2) = \pi_6(S^3) = \mathbf{Z}_{12}$. Then $k_8 \in H^8(X_6; \mathbf{Z}_{12})$ is the image of the fundamental class under the transgression, and it has the property $f_6^*(k_8) = 0$. Since $H^8(X_6; \mathbf{Z}_{12}) = H^8(X_6; \mathbf{Z}_4) \oplus H^8(X_6; \mathbf{Z}_3)$ we can decompose $k_8 = (k_8^1, k_8^2)$. Let us first compute k_8^1 . To do this we first compute $H^8(X_6; \mathbf{Z}_2)$ by using the spectral sequence of the fibration $K(\mathbf{Z}_2, 6) \rightarrow X_6 \xrightarrow{\pi_5} X_5$.

$E_2^{p,q} = H^p(X_5; \mathbf{Z}_2) \otimes H^q(K(\mathbf{Z}_2, 6); \mathbf{Z}_2)$. So, the only possible elements that could give the eight dimensional homology of X_6 are: $Sq^2(i_6)$, $a \otimes i_6$, and the eight

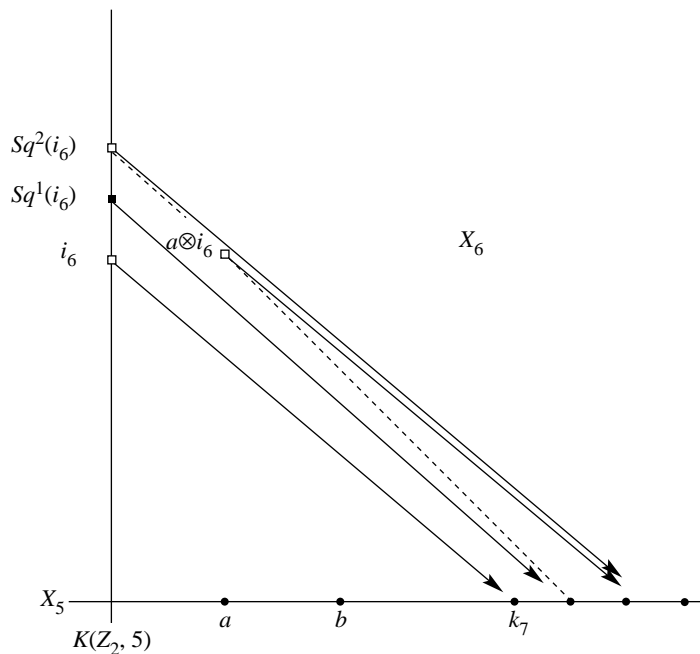


FIGURE 3

dimensional classes $\{a^2b, a^4, b^2, \alpha, \beta\}$ of X_5 . Since $d_7(i_6) = k_7$ we have $d_7(a \otimes i_6) = ak_7 \neq 0$, $a \otimes i_6$ can not contribute. Since $d_9(Sq^2(i_6)) = Sq^2(k_7)$ and $j^*(Sq^2(k_7)) = Sq^2(j^*(k_7)) = Sq^2Sq^2(i_5) = Sq^3Sq^1(i_5)$, from the above calculation, we see that $Sq^2(k_7)$ must be a nontrivial element of $H^9(X_5; \mathbf{Z}_2)$. It is also easy to check that this element $Sq^2(k_7)$ has not been killed by the previous differentials, hence $Sq^2(i_6)$ does not survive the given nontrivial cohomology class. Also $d_8(Sq^1(i_6)) = Sq^1(k_7)$, and $j^*(Sq^1(k_7)) = Sq^1(j^*k_7) = Sq^1Sq^2(i_5) = Sq^3(i_5)$ by the above calculation. $Sq^1(k_7)$ is equal to α up to an addition with classes $\{a^2b, a^4, b^2\}$. After a change of basis we can assume $Sq^1(k_7) = \alpha$. So, α does not survive the spectral sequence. After a routine check it is readily seen that the remaining classes do survive to give the homology of X_6 . So $H^8(X_6; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$, and it is generated by $\{a^2b, a^4, b^2, C\}$, such that $C = \pi_5^*(\beta)$. Figure 3 shows the nonzero differentials. Now notice that

$$\begin{aligned} j^*Sq^1(\beta) &= Sq^1j^*(\beta) = Sq^1Sq^2Sq^1(i_5) = Sq^3Sq^1(i_5) \\ &= Sq^2Sq^2(i_5) = Sq^2j^*(k_7) = j^*Sq^2(k_7). \end{aligned}$$

Hence by our previous calculations $Sq^1(\beta) = Sq^2(k_7) = \gamma$ then

$$Sq^1(C) = Sq^1\pi_5^*(\beta) = \pi_5^*Sq^1(\beta) = \pi_5^*Sq^2(k_7) = Sq^2\pi_5^*(k_7) = 0.$$

Therefore C is actually a class with \mathbf{Z}_4 coefficients and it is easily seen that $f_6^*(C) = 0$ and $C = k_8^1 \in H^8(X_6; \mathbf{Z}_4)$.

Now it remains to compute $k_8^3 \in H^8(X_6; \mathbf{Z}_3)$. Since mod 3 cohomology does not see the torsion in $K(\mathbf{Z}_2, n)$ we see that $k_8^3 = (\pi_4 \circ \pi_5)^*(s)$ for some $s \in H^8(X_4; \mathbf{Z}_3)$. We must also have $f^*(s) = 0$ in \mathbf{Z}_3 cohomology. We can easily see that $s = St_3^1(i_2) - i_2i_1^2 - i_2^2$ is the unique element satisfying this property. We can check this (as in

the case of k_4) by using the splitting principle, i.e. by assuming that the universal bundle is a direct sum of line bundles $L_1 \oplus L_2$. Then $f^*s = St_3^1(c_2) - c_2c_1^2 - c_2^2$, and the following calculation gives $f^*s = 0$.

$$\begin{aligned} St_3^1(c_2) &= St_3^1(c_1L_1.c_1L_2) = St_3^1(c_1L_1).c_1L_2 + c_1L_1.St_3^1(c_1L_2) \\ &= (c_1L_1)^3.c_1L_2 + c_1L_1.(c_1L_2)^3 = c_1L_1.c_1L_2((c_1L_1)^2 + (c_1L_2)^2) \\ &= c_2((c_1L_1 + c_1L_2)^2 - 2c_1L_1.c_1L_2) = c_2(c_1^2 - 2c_2) \pmod 3. \quad \square \end{aligned}$$

The k -invariants of $BU(2)$ gives rise to secondary cohomology operations. To compute them we need to know how they behave under pull-backs. Let $E \rightarrow BU(2)$ be the bundle obtained by pulling back $X_5 \rightarrow X_4$ by f . Since $k_6 \circ f$ is null homotopic, E can be identified by the trivial bundle

$$\begin{array}{ccc} K(\mathbf{Z}_2, 5) \times BU(2) & \xrightarrow{\tilde{f}} & X_5 \\ \downarrow & & \downarrow \\ BU(2) & \xrightarrow{f} & X_4 \end{array}$$

Let \tilde{f} be the map covering f , then we have

Lemma 3.2.

$$\tilde{f}^*(k_7) = Sq^2(i_5) \times 1 + i_5 \times c_1.$$

Proof. First of all by construction $\tilde{f}^*(k_7)$ has to be nonzero since it has to restrict to $Sq^2(i_5)$ on the fiber. By considering f and \tilde{f} as inclusions, we can apply the Thomas exact sequence of this relative fibration ([T], p. 18) to get the exact sequence:

$$\dots \rightarrow H^7(X_5; \mathbf{Z}_2) \xrightarrow{\tilde{f}^*} H^7(K(\mathbf{Z}_2, 5) \times BU(2); \mathbf{Z}_2) \xrightarrow{\tau} H^8(X_4, BU(2); \mathbf{Z}_2)$$

where τ is the relative transgression. Since $H^7(BU(2)) = 0$ τ takes values in $H^8(X_4; \mathbf{Z}_2)$. If $g : K(\mathbf{Z}_2, 5) \times BU(2) \rightarrow X_4$ is the projection followed by f , then we can define a right action of $H^*(X_4; \mathbf{Z}_2)$ on $H^*(K(\mathbf{Z}_2, 5) \times BU(2); \mathbf{Z}_2)$ by $u.v = u \smile g^*(v)$. By ([T], p. 14), the transgression is equivariant under this action i.e. $\tau(u.v) = \tau(u) \smile v$. Hence

$$\begin{aligned} \tau(i_5 \times c_1) &= \tau((i_5 \times 1) \smile (1 \times c_1)) = \tau((i_5 \times 1) \smile g^*(i_2)) \\ &= \tau(i_5 \times 1) \smile i_2 = (Sq^2(i_4) + i_2i_4)i_2 = Sq^2(i_2i_4). \end{aligned}$$

i_2 and i_4 , being integral classes, are annihilated by Sq^1 . Since $Sq^2Sq^2 = Sq^3Sq^1$ we have $Sq^2Sq^2(i_4) = 0$. Therefore

$$\tau(Sq^2(i_5 \times 1)) = Sq^2(k_6) = Sq^2(i_2i_4).$$

Therefore $Sq^2(i_5) \times 1 + i_5 \times c_1$ lies in the kernel of τ , hence by the exact sequence this must be $\tilde{f}^*(k_7)$ (recall that k_7 is the only nonzero element of $H^7(X_5; \mathbf{Z}_2)$). \square

The fibration $K(\mathbf{Z}_2, 8) \rightarrow K(\mathbf{Z}_4, 8) \xrightarrow{\rho_2} K(\mathbf{Z}_2, 8)$ allows us to break the \mathbf{Z}_4 -class k_8^1 into two \mathbf{Z}_2 classes k_8^{11} , and k_8^{12} . This means that we first kill the class $k_8^{11} := \rho_2 \circ k_8^1$ in tower construction then kill the other \mathbf{Z}_2 class that appears in the

next stage. This allows us to decompose the tower as follows:

$$\begin{array}{ccc}
 K(\mathbf{Z}_2, 8) & \xrightarrow{j} X_8 & \xrightarrow{k_8^2} K(\mathbf{Z}_3, 8) \\
 & \pi_7 \downarrow & \\
 K(\mathbf{Z}_2, 7) & \xrightarrow{j} X_7 & \xrightarrow{k_8^{12}} K(\mathbf{Z}_2, 8) \\
 & \pi_6 \downarrow & \\
 K(\mathbf{Z}_2, 6) & \xrightarrow{j} X_6 & \xrightarrow{k_8^{11}} K(\mathbf{Z}_2, 8) \\
 & \pi_5 \downarrow & \\
 K(\mathbf{Z}_2, 5) & \xrightarrow{j} X_5 & \xrightarrow{k_7} K(\mathbf{Z}_2, 7) \\
 & \pi_4 \downarrow & \\
 BU(2) & \xrightarrow{f} X_4 & \xrightarrow{k_6} K(\mathbf{Z}_2, 6)
 \end{array}$$

$k_8^{11} = r \circ \pi_5$, where $r : X_5 \rightarrow K(\mathbf{Z}_2, 8)$ with $r \circ j = Sq^2Sq^1(i_5)$, and $k_8^{12} \circ j = Sq^1(i_7)$, and rest of k_6, k_7, k_8^2 as in Proposition 3.1.

For a prime p , it is easier to construct the mod p Postnikov tower of a space Z . This means that if \mathcal{C}_p is a class of torsion abelian groups such that the order of each element is prime to p ; we construct the Postnikov tower of Z by using groups $\pi_n(Z)_{(p)} = \pi_n(Z) \bmod \mathcal{C}_p$. This gives a space $Z_{(p)}$ approximating the mod p homotopy type of Z [MT]. Let us apply this to $BU(2)$. We have $\pi_{n+1}BU(2) = \pi_n(S^3)$ for $n \geq 2$, and the smallest integer n such that $\pi_n(S^3)$ has a nonzero element of order p is $n = 2p$. Also $\pi_{2p}(S^3) = \mathbf{Z}_p \bmod \mathcal{C}_p$. So again by calling $X_4 = K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 4)$ the first stage of mod p resolution looks like

$$BU(2)_{(p)} \rightarrow X_4 \xrightarrow{k_{2p+2}} K(\mathbf{Z}_p, 2p+2).$$

We claim: $k_{2p+2} = St_p^1(i_4) - i_4s_{p-1}(i_2, i_4)$, where s_{p-1} is the Newton polynomial satisfying $s_{p-1}(\sigma_1, \sigma_2) = x_1^{p-1} + x_2^{p-1}$ with $\sigma_1 = x_1 + x_2$ and $\sigma_2 = x_1x_2$. To see this pull back i_2 and i_4 , to c_1 and c_2 . By the splitting principle assume that the universal bundle is the sum of line bundles $L = L_1 \oplus L_2$. Call $x_i = c_1(L_i)$

$$St_p^1(c_2) = St_p^1(x_1x_2) = x_1^p x_2 + x_1 x_2^p = x_1x_2(x_1^{p-1} + x_2^{p-1}) = c_2s_{p-1}(c_1, c_2).$$

Actually $\pi_{2p+s}(S^3)_{(p)} = 0$ for $s < 2p - 3$ [T]. This implies vanishing of some higher k -invariants of the mod p tower: $k_i = 0$ for $2p + 2 < i \leq 4p - 2$.

4. HOMOTOPY OF $\mathcal{B}^*(P)$

From the weak homotopy equivalence $\mathcal{B}^*(P) \simeq \text{Map}^P(X, B_{SO(3)})$ we get

$$\begin{aligned}
 \pi_k(\mathcal{B}^*) &= [S^k, s_0; \text{Map}^P(X, B_{SO(3)}), P] \\
 &= \pi_0 \text{Map}_*(S^k, \text{Map}^P(X, B_{SO(3)})) \\
 &= \pi_0 \text{Map}(S^k \times X, B_{SO(3)})^P \\
 &= [S^k \times X, B_{SO(3)}]^P.
 \end{aligned}$$

Here s_0 denotes the base point of S^k , and the exponential P above means the set of maps that restrict to P on $s_0 \times X$ and are homotopic to P on each slice $s \times X$ (by viewing view P as a map). Hence the last expression is just the isomorphism class of bundles over $S^k \times X$ which restrict to a bundle isomorphic to P on each

slice and equal to P on a particular slice:

$$\pi_k(\mathcal{B}^*) = \left\{ \begin{array}{c} \xi \\ \downarrow \\ S^k \times X \end{array} \middle| \xi|_{s_0 \times X} = P, \xi|_{s \times X} \cong P \text{ for each } s \in S^k \right\}.$$

So, $\pi_k(\mathcal{B}^*)$ is identified by the number of different bundles on $D^k \times X$ restricting to a fixed bundle P on $S^{k-1} \times X$, where P denotes the bundle obtained by pulling back P by the projection to X . The group structure is induced by the obvious way of composing two copies of $D^k \times X$ along their face $D^{k-1} \times X$.

If $H_1(X) = 0$ the bundle $P \rightarrow X$ lifts to a $U(2)$ -bundle $\bar{P} \rightarrow X$. If case $w_2(P)$ is characteristic, i.e. $w_2(P) = w_2(TX)$, $c_2(\bar{P}) \bmod 2$ is a well-defined number, i.e. it is independent of the lifting. Recall by [AMR]:

Proposition 4.1.

$$\pi_1(\mathcal{B}^*(P)) = \begin{cases} \mathbf{Z}_2 & \text{if } \begin{cases} w_2(P) = w_2(X) \\ c_2(\bar{P}) = 0 \bmod 2 \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

In this section we will compute some higher homotopy groups of $\mathcal{B}^*(P)$. Since $D^k \times X = S^{k-1} \times X \cup (k - \text{cell}) \cup (k + 2 - \text{cells}) \cup (k + 4 - \text{cell})$, there are no indeterminacies in applying the obstruction theory to count the bundles. For $i = k, k + 2, k + 4$, let $(D^k \times X)^{(i)}$ denote the i -skeleton of $D^k \times X$ and

$$\begin{aligned} B_i &= \text{the number of different bundles on } S^{k-1} \times X \cup (D^k \times X)^{(i)} \\ &\quad \text{restricting to the fixed bundle } P \text{ on } S^{k-1} \times X, \\ A_i &= \text{the number of different ways to extending a given bundle on} \\ &\quad S^{k-1} \times X \cup (D^k \times X)^{(i-2)} \text{ to } S^{k-1} \times X \cup (D^k \times X)^{(i)}. \end{aligned}$$

Let $\theta_i : B_{i-2} \rightarrow \mathcal{O}_i$ be the obstruction to extending to the i -skeleton where

$$A_i = H^i(D^k \times X, S^{k-1} \times X; \pi_i B_{SO(3)}) = H^{i-k}(X; \pi_{i-1} SO(3)),$$

$$\mathcal{O}_i = H^i(D^k \times X, S^{k-1} \times X; \pi_{i-1} B_{SO(3)}) = H^{i-k}(X; \pi_{i-2} SO(3)).$$

For example $\pi_k(\mathcal{B}^*) = B_{k+4}$. The group $\pi_k(\mathcal{B}^*)$ contains at least the “trivial” element P . B_{k+4} is a subset of $\bigoplus H^{i-k}(X; \pi_{i-1} SO(3))$. In particular, all $\pi_k(\mathcal{B}^*)$ are finite if $k > 4$. We can also identify the obstructions with the k -invariants in the Postnikov tower of $BU(2)$.

Let $\pi_k(\mathcal{B}^*)_0$ be the subgroup of bundles in $\pi_k(\mathcal{B}^*)$ which are isomorphic to P in the complement of a point in $S^k \times X$. Clearly the elements of $\pi_k(\mathcal{B}^*)_0$ are obtained from P by clutching at the top skeleton by bundles $\eta \rightarrow S^{k+4}$. Let $\pi_k(\mathcal{B}^*)_1$ be the cosets of this subgroup, i.e. it is the equivalence class of bundles which differ in the complement of a point, we have a short exact sequence

$$0 \rightarrow \pi_k(\mathcal{B}^*)_0 \rightarrow \pi_k(\mathcal{B}^*) \rightarrow \pi_k(\mathcal{B}^*)_1 \rightarrow 0.$$

$\pi_k(\mathcal{B}^*)_1$ consists of elements of B_{k+2} which are annihilated by θ_{k+4} . Clearly $\pi_k(\mathcal{B}^*)_1$ is obtained from $\pi_k(\mathcal{B}^*)_0$ by clutching at the top skeleton by bundles $\eta \rightarrow S^{k+4}$. For $\alpha \in \pi_k(\mathcal{B}^*)_0$ and the bundle corresponding to $\eta \in \pi_{k+3} SO(3)$, let $\alpha \# \eta$ denote the bundle obtained by clutching α with η in a chart. So we have the following useful fact.

Lemma 4.2. *If $\pi_{k+3}SO(3) = \mathbf{Z}_k$ and η is a generator of this group then*

$$\pi_k(\mathcal{B}^*)_0 = \begin{cases} 0 & \text{if } P\sharp\eta \cong P, \\ \mathbf{Z}_k & \text{if } P\sharp\eta^i \not\cong P \text{ for all } i = 1, \dots, k. \end{cases}$$

The following idea of Morgan gives us a useful way of expressing $\alpha\sharp\eta \cong \alpha$.

Lemma 4.3. *There is an $SO(3)$ -bundle $\xi \rightarrow S^1 \times S^k \times X - \text{int}(B^{5+k})$, such that $\xi|_{S^{4+k}} = \eta$ and $\xi|_{0 \times S^k \times X} = \alpha$ if and only if $\alpha\sharp\eta \cong \alpha$.*

Proof.

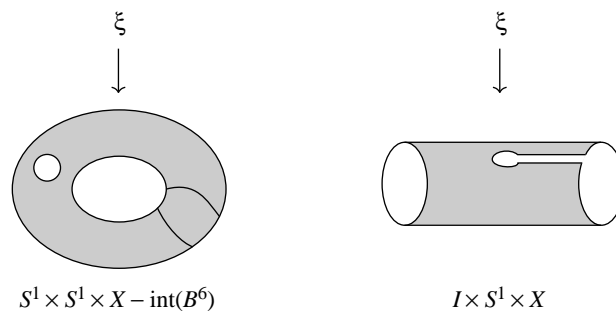


FIGURE 4

Given such a bundle ξ we cut $S^1 \times S^k \times X - \text{int}(B^{5+k})$ open to get a bundle over the punctured $I \times S^k \times X$ which restricts to α on the two outside boundary components and to η over the puncture S^5 .

Then by connecting summing S^{4+k} to one of the outside boundary components we get a bundle over $I \times X$ restricting α and $\alpha\sharp\eta$ over the two ends, hence these two bundles are isomorphic. Clearly the converse also holds. \square

Proposition 4.4. *If $\pi_1(\mathcal{B}^*(P)) = 0$, then*

$$\pi_2(\mathcal{B}^*(P)) = \begin{cases} H^2(X; \mathbf{Z}) & \text{if } w_2(P) \neq w_2(TX), \\ H^2(X; \mathbf{Z}) \oplus \mathbf{Z}_2 & \text{if } w_2(P) = w_2(TX). \end{cases}$$

Proof. We can always assume that $c_2(\bar{P})$ is odd. This is because $c_2 = \frac{1}{4}(c_1^2 - p_1)$ and, hence the changing of the lifting c_1 to $c_1 + 2\theta$ changes c_2 to $c_2 + (c_1\theta + \theta^2)$. So in case $w_2(P) \neq w_2(TX)$, by choosing θ we can change the parity of c_2 . When $w_2(P) = w_2(TX)$ we must already have c_2 odd, otherwise $\mathcal{B}^*(P)$ would not be simply connected by Proposition 4.1.

Let $\xi \rightarrow S^2 \times X$ be an element of $\pi_2(\mathcal{B}^*)$. Since the complement of a point in $S^2 \times X$ is a 4 complex, over $S^2 \times X - \text{int}(B^6)$ this bundle is determined by its first Pontryagin class $p_1(\xi)$ and the second Steifel-Whitney class $w_2(\xi)$ ([DW]). Since $S^2 \times X$ is simply connected ξ lifts to a $U(2)$ -bundle $\bar{\xi}$. By the Kuneth theorem the Chern classes of $\bar{\xi}$ can be expressed as

$$c_1(\bar{\xi}) = 1 \times c_1(\bar{P}) + \epsilon([S^2] \times 1), \quad c_2(\bar{\xi}) = 1 \times c_2(\bar{P}) + [S^2] \times \alpha,$$

where $\alpha \in H^2(X)$ and $\epsilon = 0$ or 1 , and $[S^2]$ denotes the fundamental class of S^2 .

$$w_2(\xi) = c_1(\bar{\xi}) \pmod{2},$$

$$p_1(\xi) = c_1(\bar{\xi})^2 - 4c_2(\bar{\xi}) = 1 \times p_1(P) + [S^2] \times (2\epsilon c_1(\bar{P}) - 4\alpha).$$

Since ξ is defined on all of $S^2 \times X$ the obstruction to extending to the top skeleton must vanish (notice if it extends as an $SO(3)$ -bundle then it must extend as a $U(2)$ -bundle), i.e. $\theta_6(\xi) = 0$. By definition $\theta_6(\xi)$ can be identified with the k -invariant k_6 in the Postnikov tower of $BU(2)$, so

$$k_6(\bar{\xi}) = Sq^2(c_2(\bar{\xi})) + c_1(\bar{\xi})c_2(\bar{\xi}) \pmod 2.$$

An easy calculation shows that $k_6(\bar{\xi}) = \alpha^2 + c_1\alpha + \epsilon c_2 \pmod 2$. Since c_2 is odd, the condition $k_6(\xi) = 0$ implies $\epsilon = \langle \alpha^2 + c_1\alpha, [X] \rangle \pmod 2$, i.e. ϵ is determined by α . Hence $\xi \mid S^2 \times X - \text{int}(B^6)$ is determined by $\alpha \in H^2(X)$. Conversely, given α we can construct the corresponding bundle, since the obstructions to constructing it lies in $\mathcal{O}_i = 0$ for $i = 2, 4$. So $\pi_2(\mathcal{B}^*)_1 = H^2(X; \mathbf{Z})$.

We claim that $\pi_2(\mathcal{B}^*)_0 = \mathbf{Z}_2$ or 0 according to the $w_2(P)$ characteristic or not. Since $\pi_5 SO(3) = \mathbf{Z}_2$. By Lemmas 4.2 and 4.3 it is enough to show that:

(a) Every $SO(3)$ -bundle $\eta \rightarrow S^1 \times S^2 \times X - \text{int}(B^7)$ restricting to ξ on $0 \times S^2 \times X$, extends to $S^1 \times S^2 \times X$, when $w_2(P)$ is characteristic.

(b) There is an $SO(3)$ -bundle $\eta \rightarrow S^1 \times S^2 \times X - \text{int}(B^7)$ which restricts to ξ on $0 \times S^2 \times X$ and $\eta|_{S^6}$ is nontrivial, when $w_2(P)$ is not characteristic.

Since $SO(3)$ -bundles in question all lift to $U(2)$ -bundles, it suffices to show (a) and (b) for $U(2)$ -bundles. Let $\bar{\xi} \rightarrow S^1 \times S^2 \times X$ also denote the pull-back of $\bar{\xi} \rightarrow S^2 \times X$ by projection. Consider the Postnikov tower of $BU(2)$:

$$\begin{array}{ccccc} K(\mathbf{Z}_2, 5) & = & K(\mathbf{Z}_2, 5) & = & K(\mathbf{Z}_2, 5) \\ i \downarrow & & \downarrow i & & \downarrow i \\ K(\mathbf{Z}_2, 5) \times (S^1 \times S^2 \times X) & \xrightarrow{1 \times g} & K(\mathbf{Z}_2, 5) \times BU(2) & \xrightarrow{\tilde{f}} & X_5 \xrightarrow{k_7} K(\mathbf{Z}_2, 7) \\ s \uparrow \downarrow & & \downarrow & & \downarrow \\ S^1 \times S^2 \times X & \xrightarrow{g} & BU(2) & \xrightarrow{f} & X_4 \xrightarrow{k_6} K(\mathbf{Z}_2, 6) \end{array}$$

Here $f : BU(2) \rightarrow K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 4)$ is given by (c_1, c_2) , and g is the classifying map for $\bar{\xi}$. Since $k_6 \circ f$ is null homotopic, f and $f \circ g$ induces trivial $K(\mathbf{Z}_2, 5)$ -bundles over $BU(2)$ and $S^1 \times S^2 \times X$ respectively.

In particular $f \circ g$ lifts to X_5 , and the number of different liftings are given by $H^5(S^1 \times S^2 \times X, 0 \times S^2 \times X; \mathbf{Z}_2) = H_2(S^2 \times X; \mathbf{Z}_2)$. Let $\tilde{f} : K(\mathbf{Z}_2, 5) \times BU(2) \rightarrow X_5$ be the map covering f . Then the different liftings h of $f \circ g$ can be constructed by $h = \tilde{f} \circ (1 \times g) \circ s$, where $s = (\beta, id) : S^1 \times S^2 \times X \rightarrow K(\mathbf{Z}_2, 5) \times (S^1 \times S^2 \times X)$ is a section of the trivial bundle corresponding to a map $\beta : S^1 \times S^2 \times X \rightarrow K(\mathbf{Z}_2, 5)$. By Lemma 3.2 $\tilde{f}^*(k_7) = Sq^2(i_5) + i_5 \times c_1$, hence

$$h^*(k_7) = Sq^2(\beta) + \beta \smile c_1(\bar{\xi}) \pmod 2.$$

In general we can write $\beta = [S^1] \times [S^2] \times a + \delta[S^1] \times 1 \times [X]$ which gives

$$h^*(k_7) = [S^1] \times [S^2] \times (a^2 + ac_1(\bar{P}) + \delta\epsilon[X]) \pmod 2$$

where $\epsilon = \langle \alpha^2 + c_1\alpha, [X] \rangle \pmod 2$. Hence if $w_2(P)$ is not characteristic we can find the required bundle of (b) by choosing β such that $\delta = 0$ and $a \in H^2(X)$ with $a^2 + ac_1(\bar{P}) \neq 0 \pmod 2$. This means that by modifying $\bar{\xi}$ over the 5-skeleton we can get a bundle η with the desired properties.

If $w_2(P)$ is characteristic then clearly we always have $h^*(k_7) = 0$. This says that the obstruction to extending is always zero, and (a) follows from Lemma 4.9. \square

For $\pi_1(\mathcal{B}^*) = 0$, we have

$$H^2(\mathcal{B}^*; \mathbf{Z}) = \text{Hom}(H_2(\mathcal{B}^*), \mathbf{Z}) = \text{Hom}(\pi_2(\mathcal{B}^*); \mathbf{Z}) = \text{Hom}(H^2(X), \mathbf{Z}).$$

Let $\alpha_1, \dots, \alpha_n$ be a basis of $H^2(X)$, and $\epsilon_i = \langle \alpha_i^2 + c_1 \alpha_i, [X] \rangle \pmod{2}$. Then we can explicitly describe the map $\mu : H_2(X) \rightarrow H^2(\mathcal{B}^*(P))$:

Lemma 4.5. $\mu(\theta)(\alpha_i) = 4\langle \alpha_i, \theta \rangle - 2\epsilon_i \langle w_2(P), \theta \rangle$.

Proof. Let $f_i : S^2 \rightarrow \mathcal{B}^*(P)$ correspond to $\alpha_i \in H^2(X)$. If $\xi_i \rightarrow S^2 \times X$ represents f_i then $p_1(\xi_i) = 1 \times p_1(P) + [S^2] \times (2\epsilon_i c_1(P) - 4\alpha_i)$. Then ξ_i is also the pull-back of the universal bundle ξ by $f_i \times 1 : S^2 \times X \rightarrow \mathcal{B}^*(P) \times X$, hence

$$\begin{aligned} \mu(\theta)(\alpha_i) &= -\langle p_1((f_i \times 1)^* \xi) / \theta, [S^2] \rangle \\ &= -\langle p_1(\xi_i), [S^2] \times \theta \rangle \\ &= 4\langle \alpha_i, \theta \rangle - 2\epsilon_i \langle c_1(P), \theta \rangle. \quad \square \end{aligned}$$

In particular if $a_1, a_2, \dots, a_n \in H_2(X)$ are the Poincaré duals of $\alpha_1, \dots, \alpha_n$ and $\alpha_i^* \in \text{Hom}(H^2(X))$ are the vector space duals of α_i , then we can write

$$\mu(a_j) = 4 \sum_i \langle \alpha_i \smile \alpha_j, [X] \rangle \alpha_i^* - 2 \langle w_2(P), a_j \rangle \sum_i \epsilon_i \alpha_i^* \quad \square$$

Remark. In the remaining case of $\pi_1(\mathcal{B}^*(P)) = \mathbf{Z}_2$ the above result still holds. Since we do not need the result in this generality we leave the proof as an exercise.

Proposition 4.6.

$$\pi_3(\mathcal{B}^*(P))_{(3)} = \begin{cases} 0 & \text{if } p_1(P) \not\equiv 0 \pmod{3}, \\ \mathbf{Z}_3 & \text{if } p_1(P) \equiv 0 \pmod{3}. \end{cases}$$

Proof. For $k = 3$ the sets $A_3 = 0$ and $A_5 = H^2(X; \mathbf{Z}_2)$. We are doing our calculations mod \mathcal{C} , where \mathcal{C} is the class of torsion abelian groups whose orders prime to 3. Hence there is a unique bundle $\xi \rightarrow S^3 \times X - \text{int}(B^7)$ restricting to P on $s_0 \times X$. As in the proof of Proposition 4.4, by identifying the obstruction θ_7 to extending ξ to $S^3 \times X$ by k_7 , which is 0 mod \mathcal{C} . So $\pi_3(\mathcal{B}^*)_1 = 0 \pmod{\mathcal{C}}$.

By Lemmas 4.1 and 4.2 it is enough to show that when $p_1(P) \equiv 0 \pmod{3}$ every $SO(3)$ -bundle $\eta \rightarrow S^1 \times S^3 \times X - \text{int}(B^8)$ with $\eta|_0 \times S^3 \times X = \xi$, extends to $S^1 \times S^3 \times X$; and when $p_1(P) \not\equiv 0 \pmod{3}$ there exists an $SO(3)$ -bundle η which does not extend.

Notice that $p_1(P) \equiv 0 \pmod{3}$ implies $c_1^2(P) \equiv c_2(P) \pmod{3}$. From Postnikov tower of $BU(2)$ the obstructions to extending η are given by the k -invariants (k_8^1, k_8^2) , and $k_8^1 \equiv 0 \pmod{\mathcal{C}}$. By Proposition 3.1 k_8^2 is a primary obstruction and

$$k_8^2(\eta) = s(\eta) = St_3^1 c_2(\eta) - c_2(\eta) c_1(\eta)^2 - c_2(\eta)^2.$$

By the Kunneth theorem the characteristic classes of η must look like

$$c_1(\eta) = 1 \times 1 \times c_1(P), \quad c_2(\eta) = 1 \times 1 \times c_2(P) + \lambda([S^1] \times [S^3] \times 1)$$

where $\lambda \in \mathbf{Z}$. Hence $k_8^2(\eta) = -\lambda[S^1] \times [S^3] \times (c_1(P)^2 + 2c_2(P))$. But this vanishes mod 3 when $p_1(P) \equiv 0 \pmod{3}$. \square

Proposition 4.7. *If $\pi_1(\mathcal{B}^*) = 0$ and p any odd prime, then $\pi_4(\mathcal{B}^*)_{(p)} = \mathbf{Z}$. Furthermore if $p > 3$*

$$\pi_k(\mathcal{B}^*)_{(p)} = \begin{cases} 0 & \text{if } 4 < k < 2p - 3, \\ \mathbf{Z}_p & \text{if } k = 2p - 3. \end{cases}$$

The vector space dual of the induced homology class of the generator $[S^4]$ of $\pi_4(\mathcal{B}^*)_{(p)}$ is $\mu^*(1) = -\bar{p}_1/4$ when $p > 3$ or $p = 3$ and $p_1(P) = 0 \pmod 3$, otherwise it is $\frac{1}{3}\mu^*(1)$. In particular if $p_1(P) \neq 0 \pmod 3$ then $\bar{p}_1[S^4] = 0 \pmod 3$.

Proof. For simplicity write $\pi_k(\mathcal{B}^*) = \pi_k(\mathcal{B}^*)_{(p)}$. Since $\pi_k(S^3)$ has no p -torsion for $k < 2p$ and we are computing only the p primary components, it follows from the definitions that for $p > 3$: $A_4 = \mathbf{Z}$ for $k = 4$, and $A_k = A_{k+2} = A_{k+4} = 0$ for $4 < k < 2p - 3$, and $A_{k+4} = \mathbf{Z}_p$ for $k = 2p - 3$. Also $\mathcal{O}_{k+2} = \mathcal{O}_{k+4} = 0$ for $3 < k < 2p - 2$. Now the statements $\pi_4(\mathcal{B}^*)_{(p)} = \mathbf{Z}$ and $\pi_k(\mathcal{B}^*)_{(p)} = 0$ for $4 < k < 2p - 3$, follow from obstruction theory definitions.

Furthermore we conclude that $\pi_{2p-3}(\mathcal{B}^*) = \pi_{2p-3}(\mathcal{B}^*)_0$, i.e. the only possible nontrivial elements are represented by bundles $\eta \rightarrow S^{2p-3} \times X$ obtained by twisting the “trivial” bundle P at the top cell. As before by the aid of Lemmas 4.2 and 4.3 we will conclude $\pi_{2p-3}(\mathcal{B}^*) = \mathbf{Z}_p$. This follows by proving that every bundle $\xi \rightarrow (S^1 \times S^{2p-3} \times X)_0$, which agrees with P on $0 \times S^{2p-3} \times X$, extends. We show this by writing $c_1(\xi) = 1 \times 1 \times c_1(P)$ and $c_2(\xi) = 1 \times 1 \times c_2(P)$. The relevant k -invariant measuring the obstruction to extension is k_{2p+2} . It is computed at the end of section 3 :

$$k_{2p+2}(\xi) = St_p^1(c_2) - c_2 s_{p-1}(c_1, c_2) = 0.$$

Hence by Lemma 4.9 ξ extends. In case of $p = 3$ and $k = 4$, we make the same argument, except we have $\mathcal{O}_8 = \mathbf{Z}_3$ and the obstruction to extending the elements $\lambda \in A_4 = \mathbf{Z}$ to the top skeleton is the following element of \mathbf{Z}_3 :

$$\theta_8 = \lambda[S^4] \times (c_1(P)^2 + 2c_2(P)).$$

θ vanishes when $p_1(P) = 0 \pmod 3$ or $\lambda = 0 \pmod 3$. Hence the generator λ of $\pi_4(\mathcal{B}^*)$ is identified by 1 or 3 in \mathbf{Z} according to $p_1(P) = 0 \pmod 3$ or not.

To see the last assertion let $f : S^4 \rightarrow \mathcal{B}^*$ be the generator, this corresponds to a bundle $\eta \rightarrow S^4 \times X$ such that $c_1(\eta) = 1 \times c_1(P)$ and $c_2(\eta) = 1 \times c_2(P) + \lambda([S^4] \times 1)$ with $\lambda \in \mathbf{Z}$, and hence $p_1(\eta) = 1 \times p_1(P) - 4\lambda[S^4] \times 1$. η is also the pull-back of the universal bundle ξ by $f \times 1 : S^4 \times X \rightarrow \mathcal{B}^*(P) \times X$, hence

$$\langle \bar{p}_1, f_*[S^4] \rangle = \langle p_1(\xi), f_*[S^4] \times 1 \rangle = \langle (f \times 1)^* p_1(\xi), [S^4] \times 1 \rangle = -4\lambda. \quad \square$$

Remark 4.8. Similar arguments give $\pi_5(\mathcal{B}^*)_{(3)} = H^2(X; \mathbf{Z}_3)$, and for $p > 3$

$$\pi_k(\mathcal{B}^*)_{(p)} = \begin{cases} H^2(X, \mathbf{Z}_p) & \text{if } k = 2p - 1, \\ \mathbf{Z}_p & \text{if } k = 2p + 1, \\ 0 & \text{for other } 2p - 3 < k < 4p - 6. \end{cases}$$

If the obstructions to extending a bundle ξ from $(k - 1)$ -skeleton to the k skeleton of a CW-complex vanishes, then we can extend ξ to the k -skeleton by possibly readjusting it over the $k - 1$ cells. The following lemma says that in the cases we are interested in we can extend it without readjusting the bundle.

Lemma 4.9. *Let $\xi \rightarrow (S^1 \times S^n \times X)_0$ be an $SO(3)$ -bundle over the punctured $S^1 \times S^n \times X$, where $n = 2, 3$ or $2p - 3$ for an odd prime p . For $n = 2$ also assume that $w_2(\xi)(1 \times S^2 \times 1) = 0$. Then this bundle extends to $S^1 \times S^n \times X$, if the obstruction k_{n+5} to extending to the top skeleton vanishes.*

Proof. Let $Z = S^1 \times S^n \times X$ and $Z_0 = Z - \text{int}(B^{n+5})$. When $k_{n+5} = 0$ by obstruction theory ξ can be extended to Z , after possibly changing it over the $n + 4$ cells. We claim that this readjustment over $n + 4$ cells is not necessary, i.e. if ξ does not

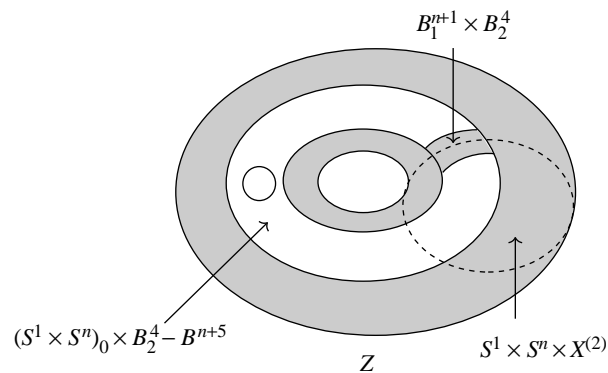


FIGURE 5

extend, then it can not be extended even after readjusting it over the $n + 4$ cells. Here we use a trick of [AMR]. If this is not the case we would have two bundles over the complement of the $(n + 3)$ -skeleton (Figure 5):

$$Z_0 - S^1 \times S^n \times X^{(2)} \cup \text{int}(B_1^{n+1}) \times B_2^4 = (S^1 \times S^n)_0 \times B_2^4 - \text{int}(B^{n+5})$$

agreeing over the outside boundary, and one extending the other not extending over B^{n+5} . Here $X^{(2)}$ is the two-skeleton of X , $B_2^4 = X - X^{(2)}$, B_1^{n+1} is a ball in $S^1 \times S^n$, and $(S^1 \times S^n)_0$ is the complement of this ball.

By putting these bundles together on the double of this manifold (two copies of the manifold glued along the common boundary) we obtain a bundle on

$$W = \text{Double}((S^1 \times S^n)_0 \times B^4) - \text{int}(B^{n+5})$$

which is nontrivial on the boundary. $W = S^1 \times S^{n+4} \# S^n \times S^5 - \text{int}(B^{n+5})$. Hence we get a map $f : W \rightarrow B_{SO(3)}$ which restricts to the nontrivial element of $\pi_{n+4}(B_{SO(3)}) = \pi_{n+3}(S^3)$ on the boundary S^{n+4} . By surgery we can turn f into a map from a $f : W^{n+5} \rightarrow B_{SO(3)}$ where W^{n+5} is a ball (in case $n = 2p - 3$ it is a \mathbf{Z}_p homology ball), contradicting the fact that f is essential on $\partial W = S^{n+4}$. \square

5. UNIVERSAL RELATIONS

Let $P \rightarrow X$ be a $SO(3)$ -bundle over a smooth Riemannian 4-manifold (X, g) . The moduli space $\mathcal{M}(P)$ of ASD connections on P is defined to be

$$\mathcal{M}(P) = \{[A] \in \mathcal{B}^*(P) \mid F_+(A) = 0\}$$

where $F_+(A)$ is the self-dual part of the curvature of the connection A with respect to metric g . For generic metrics this is a smooth oriented manifold. For a fixed w_2 , it is customary to call $k = -p_1(P)/4$ and denote $\mathcal{M}_k = \mathcal{M}(P)$. In particular, for $SU(2)$ -bundles we have $k = c_2(P)$. Taubes' gluing construction gives an imbedding representing the gluing parameter:

$$SO(3) \subset \mathcal{B}^*(P).$$

Proposition 5.1. *Let $\gamma \in \pi_3(\mathcal{B}^*(P))$ be the map given by composing the 2-fold covering map with the inclusion $SO(3) \rightarrow \mathcal{B}^*(P)$. Then γ represents the generator of $\pi_3(\mathcal{B}^*(P))_{(3)}$.*

Proof. By the exponential principle the homotopy class of the inclusion:

$$S^3 \longrightarrow \mathcal{B}^*(P) = \text{Map}^P(X, B_{SO(3)})$$

describes an $SO(3)$ -bundle $\eta \longrightarrow S^3 \times X$. By [AMR] this bundle is described by the clutching map $\Phi = \pi \circ \phi$, where $\phi : S^3 \times S^3 \rightarrow S^3$, $\phi(\rho, x) = x\rho x^{-1}$ and $\pi : S^3 \rightarrow SO(3)$ is the covering map. Since the restriction of ϕ maps $\phi : S^2 \times S^3 \rightarrow S^2$ it comes from the induced map $\tilde{\phi} : S^2 * S^3 = S^6 \rightarrow S^3$. This is the usual Hopf construction applied to ϕ , and it gives the (Blakers-Massey) generator of $\pi_6(S^6) = \mathbf{Z}_{12}$ (e.g. [J]). \square

Recall that [DK] $\dim(\mathcal{M}_k) = 8k - 3(1 + b_2^+)$, where $b_2^+ = b_2^+(X)$ is the dimension of the positive definite part of the intersection form of X . In particular when $b_2^+(X)$ is odd then $\dim(\mathcal{M}_k) = 2d$, for an integer $d := d_k = 4k - \frac{3}{2}(1 + b_2^+)$.

For $a_1, \dots, a_d \in \mathcal{A}_2(X) = \{a \in H_2(X) \mid \langle w_2(P), a \rangle = 0\}$ the Donaldson invariant is defined to be the intersection number:

$$\Phi_d(a_1, \dots, a_d) = \mathcal{M}_k \cap V_1 \cap V_2 \cap \dots \cap V_d \in \mathbf{Z}$$

where V_1, \dots, V_d are the codimension two divisors representing the duals of $\mu^*(a_1), \dots, \mu^*(a_d)$, and

$$\mu^* : \mathcal{A}_2(X) \rightarrow H^2(\mathcal{B}^*(P); \mathbf{Z})$$

is the map defined by taking slant product with $-\frac{1}{4}p_1(\xi)$ (see section 2). In general for $a_1, \dots, a_d \in H_2(X)$, we take V_i to be the divisors corresponding to the integral classes $\mu^*(2a_i)$ and count the intersection number and divide by 2^d . So in general:

$$\Phi_d(a_1, \dots, a_d) \in \mathbf{Z}[\frac{1}{2}].$$

By using the slant product map $\mu^* : H_0(X) \rightarrow H^4(\mathcal{B}^*(P); \mathbf{Z})$ we also have

$$\Phi_d(a_1, \dots, a_{d-2}, 1) = \langle \mu(1), \mathcal{M}_k \cap V_1 \cap V_2 \cap \dots \cap V_{d-2} \rangle.$$

Recall that [KM] a manifold X is called **simple** if

$$\Phi_{d+4}(a_1, \dots, a_d, 1, 1) = 4\Phi_d(a_1, \dots, a_d)$$

By convention it is usually defined: $\Phi_{d+2}(a_1, \dots, a_d, 1) = 2\Phi_d(a_1, \dots, a_d)$.

Now assume $\pi_1(\mathcal{B}^*) = 0$, then $H^2(\mathcal{B}^*) = \text{Hom}(H^2(X))$. Let b_1, \dots, b_n be a basis of $H_2(X)$ and $\beta_1, \dots, \beta_n \in H^2(X)$ be the dual basis with the intersection form $B_{ij} = \langle \beta_i \smile \beta_j, [X] \rangle$. Then if β_i^* are the vector space duals of β_i in $\text{Hom}(H^2(X))$, by Lemma 4.5 and remarks following it we have

$$\mu(2b_j) = 8 \sum_r B_{rj} \beta_r^*.$$

From this we can easily calculate that

$$\Omega = 4 \sum B_{rs} \beta_r^* \smile \beta_s^*.$$

We want to compute some homology groups of $\mathcal{B}^*(P)$, for this we need to construct its Postnikov tower. We will compute a few stages of the mod p Postnikov tower of B^* , where p is an odd prime. Let \mathcal{C}_p be a class of torsion abelian groups such that the order of each element is prime to p . The vector space duals $\{\beta_i^*\}$ of $\{\beta_i\}$ determine a map $f : B^*(P) \rightarrow Y_2$, where $Y_2 = \prod K(\mathbf{Z}, 2)$ is an n -fold product, where $n = \text{rank } H^2(X)$. The map f induces isomorphisms on π_i for $i = 1, 2$. Let F_3 be the fiber of f .

Let us first treat the case $p = 3$. If $p_1(P) = 0 \pmod 3$, we have $\pi_3(B^*) = \mathbf{Z}_3 \pmod{\mathcal{C}_p}$ and $H^3(F_3; \mathbf{Z}_3) = \mathbf{Z}_3$. The first k -invariant k_4 is the transgression of the fundamental class $\tau(i_3)$ in the Serre exact sequence:

$$\dots \rightarrow H^3(F_3; \mathbf{Z}_3) \xrightarrow{\tau} H^4(Y_2; \mathbf{Z}_3) \xrightarrow{f^*} H^4(B^*; \mathbf{Z}_3) \rightarrow \dots$$

Hence k_4 is the unique element in the kernel of f^* . Since $p_1(P) = 0 \pmod 3$, by Proposition 2.2 and Corollary 2.3 class $\Omega \in H^4(B^*; \mathbf{Z})$ is divisible by 3. We see that Ω is the pull back of the class $\alpha = \sum_{ij} B_{rs} \gamma_r \smile \gamma_s \in H^4(Y_2)$ by f , where $\gamma_r = 1 \times \dots \times i_2 \times \dots \times 1 \in H^2(Y_2)$ are the standard generators (here i_2 occurring in the r th slot). Since (B_{ij}) is unimodular intersection form, the \mathbf{Z}_3 -reduction $k_4 \in H^4(Y_2; \mathbf{Z}_3)$ of α is a nonzero class. Also it lies in the kernel of f^* , where f^* is the map induced in cohomology with \mathbf{Z}_3 coefficients. So k_4 is the first k -invariant. Let Y_3 be the pull-back fibration by k_4 , and Y_4 by k_5 :

$$\begin{array}{ccc} & Y_4 & \\ & \downarrow & \\ & Y_3 & \xrightarrow{k_5} K(\mathbf{Z}, 5) \\ & \downarrow & \\ \mathcal{B}^*(P) & \xrightarrow{f} Y_2 & \xrightarrow{k_4} K(\mathbf{Z}_3, 4) \end{array}$$

In case $p_1(P) \neq 0 \pmod 3$ we have $\pi_3(B^*) = 0 \pmod{\mathcal{C}}$. So in this case $k_4 = 0$ and hence $Y_3 = Y_2$. By a simple spectral sequence argument we see that $H^5(Y_3) = 0$, hence $k_5 = 0$. Therefore the fourth stage approximation of $\mathcal{B}_{(3)}^*$ is:

$$Y_4 = \begin{cases} Y_3 \times K(\mathbf{Z}, 4) & \text{when } p_1(P) = 0 \pmod 3, \\ Y_2 \times K(\mathbf{Z}, 4) & \text{when } p_1(P) \neq 0 \pmod 3. \end{cases}$$

Similarly, by using Proposition 4.7 and Remark 4.8 we can construct the Postnikov resolution of $\mathcal{B}_{(p)}^*$ for an odd prime $p > 3$. From $\pi_{2p-3}(\mathcal{B}^*) = \mathbf{Z}_p$ we get the following nontrivial first stage Postnikov tower of $\mathcal{B}^*(P)_{(p)}$:

$$\mathcal{B}_{(p)}^* \xrightarrow{F} \prod K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 4) \xrightarrow{k_{2p-2}} K(\mathbf{Z}_p, 2p-2)$$

where $F = (f, -\bar{p}_1)$. Let $Y_{2p-4} = \prod K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 4)$. In case $w_2(P) = 0$, by Proposition 2.4 we can identify k_{2p-2} as follows: If $\gamma_i \in H^2(Y_{2p-4}; \mathbf{Z}_p)$, $i = 1, \dots, n$, are the two dimensional generators and $\rho \in H^4(Y_{2p-4}; \mathbf{Z}_p)$ is the four dimensional generator, and $p = 2m + 1$ and $p_1 = \langle p_1(P), [X] \rangle$ then

$$k_{2p-2} = 4 \left(\sum B_{ij} \gamma_i \smile \gamma_j \right) \smile \rho^{m-1} - p_1 \rho^m$$

in particular:

$$F^*(k_{2p-2}) = \Omega \smile \bar{p}_1^{m-1} - p_1 \bar{p}_1^m$$

where $\bar{p}_1 = -\mu(1)$ is the Pontryagin class of the base point fibration. Recall as an integral class $F^*(k_{2p-2}) = p \theta_{2p-2}$, for some $\theta_{2p-2} \in H^{2p-2}(\mathcal{B}^*)$.

Let Y_{2p-3} be the pull-back of the universal fibration by k_{2p-2} over Y_{2p-4} . Since $\pi_{2p-2}(\mathcal{B}^*) = 0$ the $2p - 2$ stage approximation Y_{2p-2} of \mathcal{B}^* is equal to Y_{2p-3} . The spectral sequence of the fibration $K(\mathbf{Z}_p, 2p-3) \rightarrow Y_{2p-3} \rightarrow Y_{2p-4}$ immediately gives $H^{2p-2}(\mathcal{B}^*)$ torsion free (mod \mathcal{C}_p), and

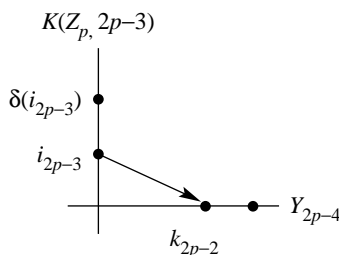


FIGURE 6

Proposition 5.2. *Let $P \rightarrow X$ be an $SU(2)$ -bundle, $p = 2m + 1$ an odd prime and $k < 2p - 2$, then $H^k(\mathcal{B}^*; \mathbf{Z}_p) = H^k(Y_{2p-4}; \mathbf{Z}_p)$. Furthermore assuming $p_1(P) = 0 \pmod{3}$ in case $p = 3$, we have a mod \mathcal{C}_p short exact sequence:*

$$0 \rightarrow H^{2p-2}(Y_{2p-4}) \xrightarrow{F^*} H^{2p-2}(\mathcal{B}^*) \rightarrow \mathbf{Z}_p \rightarrow 0,$$

$$0 \rightarrow H^{2p-2}(Y_{2p-4}; \mathbf{Z}_p) / \langle k_{2p-2} \rangle \xrightarrow{F^*} H^{2p-2}(\mathcal{B}^*; \mathbf{Z}_p) \rightarrow \mathbf{Z}_p \rightarrow 0.$$

In the spectral sequence of $K(\mathbf{Z}_p, 2p - 3) \rightarrow Y_{2p-3} \rightarrow Y_{2p-4}$ the fundamental class i_{2p-3} transgresses to the class k_{2p-2} . So the differential of the mod p Bockstein $\delta(i_{2p-3})$ maps to $\delta(k_{2p-2})$. Since k_{2p-2} comes from an integral class $\delta(k_{2p-2}) = 0$, so $\delta(i_{2p-3})$ can be viewed as a class in $H^{2p-2}(Y_{2p-3}; \mathbf{Z}_p)$. In fact if $F : \mathcal{B}^* \rightarrow Y_{2p-3}$ is the approximation map then $F^* \delta(i_{2p-3}) = \theta_{2p-2}$. (See Figure 6.)

Let Y_k^p to be the k th stage Postnikov approximation of $\mathcal{B}_{(p)}^*$. By continuing as above (by using Remark 4.8), we get

$$Y_5^3 = Y_4 \times_{k_6} \prod K(\mathbf{Z}_3, 5),$$

$$Y_{4p-7}^p = Y_{2p-2} \times_{k_{2p}} \prod K(\mathbf{Z}_p, 2p - 1) \times_{k_{2p+2}} K(\mathbf{Z}_p, 2p + 1), \quad \text{for } p > 3.$$

The k -invariants k_{2p} and k_{2p+2} come from classes in $H^*(Y_{2p-2}; \mathbf{Z}_p)$ and $H^*(Y_{2p}; \mathbf{Z}_p)$. Similarly, they can be calculated from the divisibility information on classes in $H^*(\mathcal{B}^*)$. For example, k_{2p} is either zero, or else it is given by $\delta(i_{2p-3}) \smile \gamma_i, i = 1, 2, \dots$, or classes in $H^{2p}(Y_{2p-4})$. To decide which, we need to know whether the classes $\theta_{2p-2} \smile \beta_i^*$, or the ones coming from $H^{2p}(Y_{2p-4})$ in $H^{2p}(\mathcal{B}^*)$ are divisible by p . Notice that vanishing of any of these k -invariants creates torsion class in $H^*(\mathcal{B}^*)$.

Theorem 5.3. *If $P \rightarrow X$ is an $SO(3)$ -bundle over a closed smooth simply connected 4-manifold, assume that $c_2(P)$ is even when $w_2(P) = w_2(X)$. Then when $d = 2 \pmod{3}$ the following holds: For any basis b_1, \dots, b_n of $H_2(X)$*

$$\sum_{i < j} q(a_i, a_j) \Phi_d(a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{d+2})$$

$$= \pm \sum_{i, j} \bar{q}(b_i, b_j) \Phi_{d+4}(a_1, \dots, a_{d+2}, b_i, b_j) \pmod{3}$$

where $(\bar{q}(b_i, b_j))$ denotes the inverse of the intersection matrix $(q(b_i, b_j))$.

Proof. $\dim(\mathcal{M}_{k+1}) = 2d_{k+1} = \dim(\mathcal{M}_k) + 8 = 2d_k + 8$. Choose surfaces $\Sigma_1, \dots, \Sigma_{d+2}$, in general position representing a_1, \dots, a_{d+2} . Let V_i be the corresponding

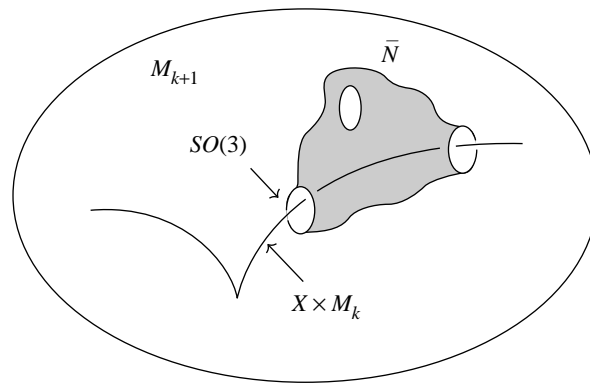


FIGURE 7

divisors that are transverse to each other and to \mathcal{M}_k and \mathcal{M}_{k+1} . This gives an oriented 4-manifold:

$$N = V_1 \cap \cdots \cap V_{d+2} \cap \mathcal{M}_{k+1}$$

in $\mathcal{B}^*(P_{k+1})$, where $P_{k+1} \rightarrow X$ is the bundle with $p_1(P_{k+1}) = p_1(P) - 4$ and $w_2(P_{k+1}) = w_2(P)$. Also recall $d_k = d$ and $2d = -2p_1 - 3(1 + b_2^+)$. (See Figure 7.)

In general N is not compact. The ends of N compactify to give cones on $SO(3)$. By [R] each end corresponds to a pair $([A], x)$, where

$$[A] \in V_1 \cap \cdots \cap \hat{V}_i \cap \cdots \cap \hat{V}_j \cap \cdots \cap V_{d+2} \cap \mathcal{M}_k \quad \text{and} \quad x \in \Sigma_i \cap \Sigma_j$$

and the number of such pairs are measured by the number:

$$(7) \quad q(\Phi_d)(\alpha_1, \dots, \alpha_{d+2}) : = \sum_{i < j} q(a_i, a_j) \Phi_d(a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{d+2}).$$

By cutting the cones we get a compact manifold with boundary $\bar{N} \subset \mathcal{B}^*(P_{k+1})$, such that each component of the boundary of \bar{N} is the gluing parameter $SO(3)$.

Since $p_1(P) = -d = -2 \pmod{3}$, we have $p_1(P_{k+1}) = 0 \pmod{3}$. Hence Propositions 4.1 and 4.6 imply $\pi_1(\mathcal{B}^*(P)) = 0$ and $\pi_3(\mathcal{B}^*(P))_{(3)} = \mathbf{Z}_3$. By Proposition 5.1 the two-fold covering map followed by the gluing parameter $S^3 \rightarrow SO(3) \subset \mathcal{B}^*(P)$ gives a generator of \mathbf{Z}_3 in the third homotopy group. So, each boundary component of \bar{N} maps to a fixed generator of \mathbf{Z}_3 . In particular the number $q(\Phi_d)(\alpha_1, \dots, \alpha_{d+2}) \pmod{3}$ represents the $(\pmod{3})$ obstruction to extending the imbedding $\bar{N} \hookrightarrow \mathcal{B}^*(P)$ to the space N^* obtained by coning each boundary component $SO(3)$ of N .

We will now compute this obstruction from the mod 3 Postnikov tower of $\mathcal{B}^*(P)$. Equating these two ways of computing the obstruction will give us the required formula. Since $p_1(P_{k+1}) = 0 \pmod{3}$, the first k -invariant is $k_4 = \sum_{ij} B_{rs} \gamma_r \smile \gamma_s \in H^4(Y_2; \mathbf{Z}_3)$, where (B_{ij}) is the intersection form. Since N is orientable and \mathcal{B}^* is simply connected, after doing some ambient surgeries in \mathcal{B}^* , we can assume that it is simply connected. Now it is clear that the only obstruction to extending the inclusion $i : \bar{N} \hookrightarrow \mathcal{B}^*$ to a map from its compactification $N^* \hookrightarrow \mathcal{B}^*$ is given by the class $k_4 \circ j \in H^4(N^*; \mathbf{Z}_3) = \mathbf{Z}_3$, where f is the first map of the Postnikov tower and $j : N^* \rightarrow Y_2$ is the extension of $f \circ i$. We can compute this by evaluating $k_4 \circ f$ on \bar{N} :

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & Y_3 & \xrightarrow{k_5} & K(\mathbf{Z}, 5) \\
 & & \downarrow & & \\
 \mathcal{B}^*(P) & \xrightarrow{f} & Y_2 & \xrightarrow{k_4} & K(\mathbf{Z}_3, 4) \\
 \uparrow i & & \uparrow j & & \\
 \bar{N} & \hookrightarrow & N^* & &
 \end{array}$$

Hence the obstruction is obtained by evaluating k_4 on \bar{N} . This is done by taking codimension two divisors $V'_i \subset \mathcal{B}^*$ representing $\mu^*(b_i)$ and computing the intersection numbers with orientations

$$\frac{1}{4} \sum_{i,j} \bar{q}(b_i, b_j) N \cap V'_i \cap V'_j = \sum_{i,j} \bar{q}(b_i, b_j) N \cap V'_i \cap V'_j \pmod{3}.$$

This gives the other term in our identity. □

By the notations suggested in [R]

$$(8) \quad q(\Phi_d)(a_1, \dots, a_{d+2}) = \sum_{i < j} q(a_i, a_j) \Phi_d(a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{d+2}),$$

$$(9) \quad i_q(\Phi_{d+4})(a_1, \dots, a_{d+2}) = \sum_{i,j} \bar{q}(b_i, b_j) \Phi_{d+4}(a_1, \dots, a_{d+2}, b_i, b_j).$$

we can express the conclusions of Theorem 5.3 as

$$(10) \quad q(\Phi_d) = i_q(\Phi_{d+4}) \pmod{3}$$

i.e. the exterior product of Φ_d with the intersection form is equal to the interior product of Φ_{d+4} with the intersection form (contraction). We can similarly denote the evaluation of the Pontryagin class as an interior product:

$$(i_1 \Phi_{d+2})(a_1, \dots, a_d) = \Phi_{d+2}(a_1, \dots, a_d, 1) = 2\Phi_d(a_1, \dots, a_d).$$

For example if X is simple then $i_1^2(\Phi_{d+4}) = 4\Phi_d$. The following is a generalization of Theorem 5.2:

Theorem 5.4. *If $P \rightarrow X$ is an $SO(3)$ -bundle over a closed smooth simply connected 4-manifold, assume that $c_2(P)$ is even when $w_2(P) = w_2(X)$. Then*

$$i_q(\Phi_{d+4}) - (d + 1)\Phi_{d+2} \equiv \pm d(d - 1)q(\Phi_d) \pmod{3}.$$

Proof. $p_1 - 4 \equiv -d - 4 \equiv -(d + 1) \pmod{3}$. Hence in case $d \equiv 2 \pmod{3}$ the result follows from Theorem 5.3. Now assume $d \not\equiv 2 \pmod{3}$, then we have $p_1(P_{k+1}) = p_1 - 4 \not\equiv 0 \pmod{3}$. Hence $\pi_3(\mathcal{B}^*)_{(3)} = 0$ where $\mathcal{B}^* = \mathcal{B}^*(P_{k+1})$. Therefore the ends of \bar{N} are null homotopic, hence we can assume the imbedding $N \subset \mathcal{B}^*$ extends to the imbedding of the 4-cycle $N^* \subset \mathcal{B}^*$. Then Proposition 2.2 implies the result, since the right-hand side of the formula is zero being a multiple of $d(d - 1) \equiv 0 \pmod{3}$. □

Theorem 5.5. *If $P \rightarrow X$ is an $SU(2)$ -bundle over a closed smooth simply connected 4-manifold, let $p > 3$ be a prime with $c_2(P) = (p + 1)/2 \pmod{2}$ when X is even. Then*

$$q^{(p-1)/2} \Phi_d \equiv 0 \pmod{p}$$

implies

$$i_q i_1^{(p-3)/2}(\Phi_{d+2p-2}) + [d + (3b_2^+ - 1)/2] i_1^{(p-1)/2} \Phi_{d+2p-2} \equiv 0 \pmod{p}.$$

Proof. Call $c_2(P) = k$ and $p = 2m + 1$, and pick $a_1, \dots, a_{d+2m} \in H_2(X)$. Consider the bundle $P_{k+m} \rightarrow X$ with Chern class $c_2(P_{k+m}) = k + m$. Hence by hypothesis $c_2(P_{k+m})$ is odd and we have the (mod p) congruences:

$$p_1(P_{k+4m}) = -4(k + m) = -d - \frac{3}{2}(1 + b_2^+) - 4m \equiv -d - \frac{1}{2}(3b_2^+ - 1) \pmod{p}.$$

As in the proof of Theorem 5.3 consider the smooth $4m$ dimensional manifold $N \subset \mathcal{B}^*(P_{k+m})$:

$$N = V_1 \cap \dots \cap V_{d+2m} \cap \mathcal{M}_{k+m}$$

where $\dim(\mathcal{M}_{k+m}) = \dim(\mathcal{M}_k) + 8m = 2d + 8m$ and V_1, \dots, V_{d+2m} are the divisors corresponding to a_1, \dots, a_{d+2m} . Let N^* be the Uhlenbeck compactification of N , which is a compact stratified space with singular strata

$$N^* - N = \bigcup L_s.$$

Each singular stratum L_s is a smooth manifold with the $4s - 1$ dimensional link $\Sigma_s = SO(3) * SO(3) * \dots * SO(3)$ consisting of the join of s copies of $SO(3)$ for some $1 \leq s \leq m$. So $\pi_*(\Sigma_s)_{(p)} = \pi_*(S^{4s-1})_{(p)}$. Since $2p - 3 = 4m - 1$ by Proposition 4.7 $\pi_i(\mathcal{B}^*)_{(p)} = 0$ for $4 < i < 4m - 1$. Hence by obstruction theory we can assume that the imbedding $N \subset \mathcal{B}^*(P_{k+m})$ extends to the imbedding $N^* - L_m \subset \mathcal{B}^*(P_{k+m})$ in the complement of a finite set of points L_m (since we are localizing at p we might have to change N^* by a relatively prime p multiple of itself). Similar to (7) the number of points of L_m is given by

$$\#L_m \equiv (q^m \Phi_d)(a_1, \dots, a_{d+2}).$$

So the mod p obstruction to extending the imbedding $N^* - L_m \subset \mathcal{B}^*(P_{k+m})$ to $N^* \subset \mathcal{B}^*(P_{k+m})$ is

$$\#L_m[\Sigma_s] \in \pi_{4m-1}(\mathcal{B}^*)_{(p)} = \mathbf{Z}_p.$$

This obstruction can also be computed by evaluating the k -invariant on N :

$$\langle k_{2p-2}, N \rangle = \langle \Omega \smile \bar{p}_1^{m-1} - p_1(P_{k+4m}) \bar{p}_1^m, N \rangle.$$

This gives the other term in the statement of the theorem. \square

Corollary 5.6. *If $P \rightarrow X$ is an $SU(2)$ -bundle on a closed smooth simply connected simple 4-manifold, let $p > 3$ be a prime with $c_2(P) = (p+1)/2 \pmod{2}$ when X is even. Then we have the following implication of (mod p) congruences:*

$$q^{(p-1)/2} \Phi_d \equiv 0 \implies i_q(\Phi_{d+p+1}) + (2d + 3b_2^+ - 1) \Phi_{d+p-1} \equiv 0.$$

Remark. If we define $\hat{\Phi}_d : H_2(X) \rightarrow \mathbf{Z}$ by $\hat{\Phi}_d(a) = \Phi_d(a, \dots, a)$ we get

$$(q^m \hat{\Phi}_d)(a) = q(a, a)^m \frac{(d+2m)!}{2^m d!} \hat{\Phi}_d(a) = \frac{q(a, a)^m}{2^m} \prod_{i=1}^{p-1} (d+i) \hat{\Phi}_d(a).$$

In particular if $p = 2m + 1$ and $d \not\equiv 0 \pmod{p}$, then $q^m \hat{\Phi}_d = 0$. So in Theorem 5.5 and Corollary 5.6 if we use the polynomials $\hat{\Phi}_d$ instead of Φ_d we can replace the condition $q^m \hat{\Phi}_d = 0$ by $d \not\equiv 0 \pmod{p}$ in particular:

Corollary 5.7. *Let $P \rightarrow X$ be as in Corollary 5.6, then for $d \not\equiv 0 \pmod{p}$:*

$$i_q(\hat{\Phi}_{d+p+1}) + (2d + 3b_2^+ - 1) \hat{\Phi}_{d+p-1} \equiv 0 \pmod{p}.$$

If $P \rightarrow X$ is as in Theorem 5.4, then we get a stronger conclusion:

$$i_q(\hat{\Phi}_{d+4}) + (2d + 3b_2^+ - 1) \hat{\Phi}_{d+2} \equiv 0 \pmod{3}.$$

Remark. We can not determine if $\pi_1(\mathcal{B}^*)$ acts trivially on its higher homotopy groups. If this is the case, one can still do the Postnikov decomposition of \mathcal{B}^* . In particular, in this case we can remove the conditions on $c_2(P)$ in all the above results.

6. SOME APPLICATIONS

Even though by in large, the Donaldson polynomial invariants of complex surfaces can be computed by other means, the following application demonstrates use of previous theorems, which could apply more general manifolds. Let $d = d_k$ (and hence $d_{k+1} = d + 4$). As in [R] by a routine calculation we get

$$(11) \quad (2d + n)\Phi_d = i_q \circ q(\Phi_d) - q \circ i_q(\Phi_d).$$

Invariants $\Phi_d(a_1, \dots, a_d)$ are called polynomial invariants since they are elements of the ring of d -linear symmetric functions $\text{Sym}^d(H_2(X))$. In terms of a basis $\{b_j\}$ of $H_2(X)$ we can write

$$\Phi_d = \sum \Phi_d(b_{i_1}, \dots, b_{i_d})b_{i_1}^*, \dots, b_{i_d}^*.$$

For example the intersection form $Q = \sum q(b_i, b_j)b_i^*b_j^*$ is an element of $\text{Sym}^2(H_2(X))$. Also any element $\beta \in H^2(X)$ can be considered to be an element of $\text{Sym}^1(H_2(X))$. By adapting the conventions of [FS] we define the symmetric product $\gamma_1\gamma_2 \in \text{Sym}^{d_1+d_2}(H_2(X))$ of two elements $\gamma_i \in \text{Sym}^{d_i}(H_2(X))$, $i = 1, 2$, by

$$\gamma_1\gamma_2(a_1, \dots, a_{d_1+d_2}) = \frac{1}{d_1!d_2!} \sum_{\sigma \in S_{d_1+d_2}} \gamma_1(a_{\sigma(1)}, \dots, a_{\sigma(d_1)})\gamma_2(a_{\sigma(d_1+1)}, \dots, a_{\sigma(d_1+d_2)})$$

where $S_{d_1+d_2}$ is the symmetric group on $d_1 + d_2$ letters. For $\gamma \in \text{Sym}^d(H_2(X))$ by convention $\gamma^{(p)} \in \text{Sym}^{2p}(H_2(X))$ is defined by

$$\gamma^{(p)} = \frac{1}{p!}\gamma^p.$$

For complex surfaces with “large diffeomorphism group” these polynomials are known to be in the form [FM]:

$$\Phi_d = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} m_{dj}Q^{(j)}K^{(d-2j)} = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \tilde{m}_{dj}Q^jK^{d-2j}$$

where $K = c_1(X)$. Here $m_{dj} \in \mathbf{Z}$ and $\tilde{m}_{dj} = m_{dj}/j!(d - 2j)!$. Since $q(Q^j) = QQ^j = Q^{j+1}$

$$(12) \quad q(\Phi_d) = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \tilde{m}_{dj}Q^{j+1}K^{d-2j},$$

$$\Phi_{d+4} = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor + 2} \tilde{m}_{d+4,j}Q^jK^{d+4-2j},$$

$$(13) \quad i_q(\Phi_{d+4}) = \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor + 2} \tilde{m}_{d+4,j}i_q(Q^jK^{d+4-2j}).$$

By using (11) we get

$$\begin{aligned} i_q(Q^j K^{d+4-2j}) &= i_q \circ q(Q^{j-1} K^{d+4-2j}) \\ &= q \circ i_q(Q^{j-1} K^{d+4-2j}) + [2(d+2) + n]Q^{j-1} K^{d+4-2j}. \end{aligned}$$

By iteration we get

$$\begin{aligned} i_q(Q^{\frac{d}{2}+2}) &= (\frac{d}{2} + 2)(d+2+n) Q^{\frac{d}{2}+1}, \quad \text{when } d \text{ is even,} \\ i_q(Q^j K^{d+4-2j}) &= Q^j \tilde{K} K^{d+2-2j} + 2j(d+3-j + \frac{n}{2})Q^{j-1} K^{d+4-2j}. \end{aligned}$$

Here we used $i_q(K^r) = \tilde{K} K^{r-2}$ where $\tilde{K} = \sum q^{ij} K(b_i)K(b_j)$, where $q^{ij} = \bar{q}(b_i, b_j)$, and $i_q(Q) = n$. Also we make the convention that $Q^{-1} = 0$ (this term occurs when $j = 0$ in the second formula). By substituting these into (13) and equating this new expression of (12) and (13) into the formula of Theorem 5.4 and comparing the corresponding coefficients we obtain the following mod 3 congruence:

$$\begin{aligned} 2(j+1)(d+2-j+n/2)\tilde{m}_{d+4,j+1} + \tilde{K}\tilde{m}_{d+4,j} - (d+1)\tilde{m}_{d+2,j} \\ \equiv \pm d(d-1)\tilde{m}_{d,j-1} \end{aligned}$$

for $j = 0, \dots, [\frac{d}{2}] + 1$. Here we make the convention $m_{d,-1} = 0$. Hence we get

Corollary 6.1. *Let X be a simply connected complex surface with a large diffeomorphism group. Let $P \rightarrow X$ be an $SO(3)$ -bundle. Assume that $c_2(P)$ even when $w_2(P) = w_2(X)$. Then $\Phi_d \bmod 3$ is determined by Φ_{d+4} or $\Phi_{d+2} \bmod 3$. Conversely, if $\tilde{K} \neq 0$, then Φ_d or $\Phi_{d+2} \bmod 3$ along with the coefficient of the highest term $Q^{[\frac{d}{2}]+2} K^{d-[\frac{d}{2}]}$ mod 3 in Φ_{d+4} determine $\Phi_{d+4} \bmod 3$.*

Since this paper was written Kronheimer and Mrowka have proved a general structure theorem for the Donaldson polynomials of simple manifolds which implies this type of corollary as a special case. Another useful application of these congruences is to show that \mathbf{CP}^2 is not a simple manifold (a fact which was previously noticed by P. Kronheimer, P. Lisca and D. Koschick). By applying Theorem 5.4 to an $SU(2)$ -bundle $P \rightarrow \mathbf{CP}^2$ we get that, for $d = 0$ or $1 \bmod 3$

$$\Phi_{d+4}(h, h, h, \dots, h) = \Phi_{d+4}(1, 1, h, \dots, h) \bmod 3$$

where $h \in \mathbf{CP}^2$ is the generator. Hence if \mathbf{CP}^2 was simple we would have

$$\Phi_{d+4}(h, h, h, \dots, h) = \Phi_d(h, \dots, h) \bmod 3.$$

This is a contradiction since for $d = 1$ the right-hand side is zero (the space of ASD connections on \mathbf{CP}^2 of charge $k = 1$ is empty [DK]), whereas by Donaldson (for the upper positive cone chamber) the left-hand side is 1 (recall that since $b_2^+(\mathbf{CP}^2) = 1$ in this case, the Donaldson polynomial depends on the chamber structure) [Mo].

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