Notes on AlgebraicTopology (v.1)

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Preface

These are the class notes from the Algebraic Topology course which I have taught over the years in MSU; many thanks to Eylem Zeliha Yıldız and Üstün Yıldırım who organized and put the informal notes in the present form. These notes are written in a terse self contained way, as it would be presented on a blackboard in a year algebraic topology course. We intentionally sacrificed generalities in favor of brevity, so that they will be fairly elementary and easy to read while touching many essential topics, as would be expected from such a course. These notes are heavily influenced by my favorite books in the subject (listed in the bibliography). I myself was introduced to algebraic topology as a graduate student, by many pleasant lectures of Edwin Spanier, Emery Thomas, and Thomas Farell, it is a pleasure to acknowledge their influence here as well.

Contents

Chapter 1

Complexes

1.1 Simplicial complexes

An abstract *simplicial complex* is a subset $K \subset \mathcal{P}(V)$ (the set of subset of V), where $V = \{v_0, v_1, \ldots, v_N\}$ is a finite set of points, satisfying the following properties:

- $\{v\} \in K$ for each $v \in V$
- If $\sigma \in K$ and $\tau \subset \sigma \Rightarrow \tau \in K$

After identifying the elements of V with the standard coordinate vectors of $\mathbb{R}^{N+1}$, we define the *geometric realization* of any $\sigma = \{w_0, \ldots, w_k\} \in K$ to be the k-simplex:

$$|\sigma| := [w_0, w_1, \ldots, w_k] = \{\sum t_i w_i \mid \sum t_i = 1\}$$

Then the *geometric realization* of K (or a polyhedron) is defined to be $|K| = \cup_{\sigma \in V} |\sigma|$; with the obvious identification topology it sits in $\mathbb{R}^{N+1}$ We define the *r-skeleton* $K^{(r)}$ of K to be the union of all simplexes of K of dimension $\leq r$. Here we won't distinguish σ from its realization $|\sigma|$, then $\tau \subset \sigma$ will correspond τ being a face of σ, which will also be denoted by $\tau \leq \sigma$. For $\sigma \in K$ define the closed and open stars of σ by:

$$St(\sigma) = \quad \{\tau \in K | \sigma \leq \tau\} \quad \text{and} \quad \mathring{St}(\sigma) = \quad \{\mathring{\tau} \in K | \sigma \leq \tau\}$$

Definition 1.1. A map between polyhedra $f : |K| \to |L|$ simplicial if

- $f(K^{(0)}) \subset L^{(0)}$
- $f(\sum t_i v_i) = \sum t_i f(v_i)$

Theorem 1.2. *(Simplicial approximation theorem) If $f : |K| \to |L|$ map between finite dimensional polyhedra, then there is a homotopy $f \simeq g$, such that g is simplicial.*

Lemma 1.3. *$v_1, ..., v_n \in K^{(0)} \implies St(v_1) \cap \cdots \cap St(v_n) = \emptyset$, unless $v_1, \ldots, v_n$ are vertices of a simplex $\sigma \in K$ and hence $St(v_1) \cap \cdots \cap St(v_n) = St(\sigma)$.*

Proof. Since $St(v_1) \cap \cdots \cap St(v_n) = \{\tau \in K \mid v_i \leq \tau \text{ for } i = 1, 2, ..n\}$ we can conclude
$\sigma =< v_1, \ldots, v_n >\leq \tau \implies St(v_1) \cap \cdots \cap St(v_n) = St(\sigma)$ □

Proof of Theorem. Let ϵ be the Lebesgue number of the covering of $|K| \subset \mathbb{R}^{N+1}$ by the open sets $\{f^{-1}(\mathring{S}t(w)) \mid w \in L^{(0)}\}$. Let K^r be a subdivision of K, so that for all $\sigma \in K^{(r)}$ we have $diam(\sigma) < \epsilon/2$. Therefore for any vertex v of K^r, $f(St(v)) \subset \mathring{S}t(w)$ for some $w \in L^{(0)}$, which we call $w = g(v)$. This defines a map $g : K^{(0)} \to L^{(0)}$ such that $f(St(v)) \subset St(g(v))$. We can extend this map to $g : K \to L$ as follows: If $\sigma =< v_1, \ldots, v_n > \in K$, then $x \in \mathring{\sigma} \implies x \in \mathring{S}t(v_i) \ \ \forall i$, hence $f(x) \in f(\mathring{S}t(v_i)) \subset St(g(v_i)) \ \ \forall i, \implies St(g(v_i)) \cap \cdots \cap St(g(v_n)) \neq \emptyset$. Hence by the above lemma $< g(v_1), \ldots, g(v_n) >$ is a simplex of L, so we can extend g linearly over σ. Since both $f(x)$ and $g(x)$ are in σ they are homotopic by the linear homotopy $(1-t)f(x) + t(g(x))$. □

1.2 CW-complexes

Here n-cell e^n means just the n-ball B^n. We say a topological space Z is obtained from X by attaching an n-cell via a map $f : S^{n-1} = \partial e^n \to X$ if Z is the identification space.

$$Z = e^n \sqcup X / x \in \partial e^n \sim f(x)$$

We call the induced map $\Phi : e^n \to Z$ the characteristic map of this attaching.

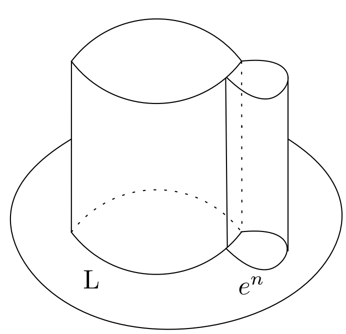
We denote $Z = X \smile_f e^n$ or simply by $Z = X + e^n$. An n-dimensional CW-complex X is a topological space which is inductively obtained from a $n-1$ dimensional CW-complex by attaching n-cells. Zero dimensional CW-complex is just a set of points. If X is a CW-complex let X^n denote the part of X consisting of cells $\leq n$. We say X is a finite CW-complex if has finite number of cells (hence it is compact). Simplicial complexes are special kind of cell-complexes where the cells are simplexes which are attached by linear maps. We call (K, L) a CW-pair if K is obtained from L by attaching cells to L. Here we will only deal with finite CW-complexes.

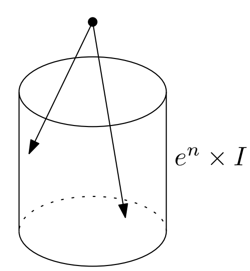
Theorem 1.4. *[HEP]*

If (K, L) CW-pair, then any homotopy $L \times I \to Y$ of a restriction of a map $K \times \{0\} \to Y$, extends to a homotopy $K \times I \to Y$.

Proof. We prove this by induction, so it suffices to assume $K = L + e^n$. Then the extension is induced by the composition of the maps, where r is the radial retraction:

$$\begin{tikzcd}
(L \times I) \sqcup e^n \times I \arrow[r, "r"] \arrow[rr, bend right] & (L \times I) \sqcup S^{n-1} \times I \sqcup e^n \times \{0\} \arrow[r, "g"] & Y
\end{tikzcd}$$





radial projection

□

Definition 1.5. For CW-complexes X, Y, a map $f \to Y$ is *cellular* if $f(X^n) \subset Y^n$.

Theorem 1.6. *(Cellular approximation theorem) For any given CW-complexes X and Y, any map $f : X \to Y$ homotopic to a cellular map; furthermore, If f is already cellular on a CW-sub complex $A \subset X$, we can assume this homotopy can be made relative to A.*

Lemma 1.7. *Let f be a map from the cell $e^n = I^n$ to CW-complex $Y + e^n$*

$$\begin{array}{ccc} f : I^n & \longrightarrow & Y \cup e^k \\ \cup & & \cup \\ f^{-1}(Y) & \dashrightarrow & Y \end{array}$$

then $f \simeq f_1$ rel $f^{-1}(Y)$ such that $\exists$ simplex $\Delta^k \subset$ interior$(e^k) = \mathbb{R}^k$ such that

i) $f_1^{-1}(\Delta^k) =$ A polyhedron K (it can be $\emptyset$)

ii) $f_1|K =$ Restriction of the linear projection

$$\begin{array}{ccccc} \overset{\circ}{I^n} & = & \mathbb{R}^n & \xrightarrow{\pi} & \mathbb{R}^k \\ & & \cup & \nearrow_{\pi|K} & \\ & & K & & \end{array}$$

Proof. Let $B_1 \subset B_2 \subset \Delta^k$ balls of radius 1 and 2 respectively. Continuity of $f|B_1$ implies uniform continuity, so when $x, y \in f^{-1}(B_2)$, $|x-y| < \epsilon \implies |f(x) - f(y)| < 1/2$. Subdivide I^n into union of cubes *diam* $< \epsilon/2$, then let K_1 be the union of closed cubes meeting $f^{-1}(B_1)$, and K_2 be the union of closed cubes meeting K_1. Hence we have

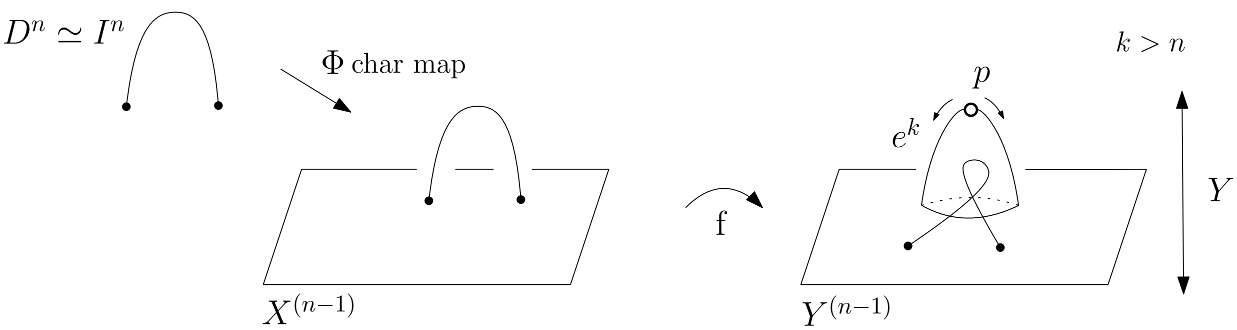
$$f^{-1}(B_1) \subset K_1 \subset K_2 \subset f^{-1}(B_2)$$

This is because every $x \in K_2$ lies in some cube δ_2 of diameter $< \epsilon/2$, which meets K_1. Therefore δ_2 meets another cube δ_1 of diameter $< \epsilon/2$ which meets $f^{-1}(B_1)$. Hence, by triangle inequality, there is $z \in f^{-1}(B_1)$ with $|z - x| < \epsilon \implies |f(z) - f(x)| < 1/2 \implies f(x) \in B_2$ so $x \in f^{-1}(B_2)$. Now by subdividing make K_1, K_2 simplicial complexes.

Now define $g : K_2 \mapsto e^k$, by letting $g|K_2^{(0)} = f|K_2^{(0)}$ then extending it linearly over the simplexes. And let $\phi : K_2 \mapsto [0,1]$ be a function such that $\phi|K_1 \equiv 1$ and $\phi|\partial K_2 \equiv 0$.

Then we can define a homotopy $f_t : K_2 \mapsto e^k$ by $f_t(x) = (1 - t\phi(x))f(x) + (t\phi(x))g(x)$ with the property $f_t|\partial K_1 = f$, $f_0 = f$ and $f_1|K_1 = g$ $\square$

Proof of Theorem. By induction, assume $f(X^{n-1}) \subset Y^{n-1}$, when we attach a $n-$cell to $n-1$ skeleton $X^{n-1} \underset{g}{\cup} e^n$. Since $f(e^n)$ compact it meets finitely many cells of Y.



By the previous Lemma, after a homotopy we can assume f is PL in the interior of the n-cell e^n, so when $k > n$, $f(e^n)$ misses center points of the k-cells e^k of Y. So we can deform $f|(X^{n-1} \cup e^n)$ by the radial homotopy from the center points of e^k to Y^{n-1}. □

For a CW-pair (K, L), we define the CW-complex obtained by collapsing the subcomplex L of K to a point by $K/L = K \cup c(L)$, where c(L) means cone on L.

Proposition 1.8. *If (K, L) a CW pair and $L \simeq * \Rightarrow K/L \simeq K$*

Proof. Since L is contractible $\exists f_t : L \to L$, $f_0 = id$, $f_1(L) = *$. By HEP (homotopy extension property) $\exists F_t : K \to K$, $F_0 = Id$, $F_1(L) = *$, therefore we can view F as a map $F_1 : K/L \to K$, more precisely $F_1 = q \circ p$ for some mapping $q : K/L \to K$. So in particular F_t provides a homotopy $q \circ\ p \simeq Id$.

$$\begin{tikzcd}
K \arrow[rr, "F_1"] \arrow[dr, "p"] & & K \\
& K/L \arrow[ur, "q"'] \arrow[ul, bend left, "q"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, "F_t"] \arrow[d, "p"'] & K \arrow[d, "p"] \\
K/L \arrow[r, "\phi_t"'] \arrow[ur, dashed, "q"] & K/L
\end{tikzcd}$$

Also since $p \circ F_t(L) = *$, we can write $p \circ F_t = \phi_t \circ p$, for some map $\phi_t : K/L \to K/L$. This gives a homotopy $p \circ q \simeq id$. □

1.3 Chain complexes and their homology

Let R be a commutative ring with unit. A *chain complex* $\mathcal{C} = \{C_k, \partial_k\}_{k\geq 0}$ is a collection of free R-modules and homomorphisms $\partial_k : C_k \to C_{k-1}$ are called boundary maps, such that $\partial_{k-1} \circ \partial_k = 0$. Define submodules of C_k, called cycles: $Z_k = ker(\partial_k)$, and boundaries: $B_k = im(\partial_{k+1})$. It follows from definition that $B_k \subset Z_k$, hence we can define the quotient $H_k(\mathcal{C}) = Z_k/B_k$ which is called the k-th homology group of $\mathcal{C}$, and the graded groups $H(\mathcal{C}; R) = \{H_k(\mathcal{C})\}_{k\geq 0}$ are called the homology of $\mathcal{C}$. Unless otherwise specified we take R to be the $\mathbb{Z}$, and we abbreviate $H(\mathcal{C}; R)$ by $H_*(\mathcal{C})$ or simply by $H(\mathcal{C})$.

A *cochain complex* $\mathcal{C}^* = \{C^k, \delta^k\}_{k\geq o}$ is a collection of free R-modules and homomorphisms $\delta_k : C^k \to C^{k+1}$ called coboundary maps, such that $\delta_{k+1} \circ \delta_k = 0$. We define cocycles to be $Z^k = ker(\delta_k)$, and coboundaries: $B^k = im(\delta_{k-1}) \implies B^k \subset Z^k$. The quotient $H^k(\mathcal{C}^*) = Z^k/B^k$ called the k-th cohomology group of $\mathcal{C}^*$, and the graded groups $H(\mathcal{C}^*; R) = \{H^k(\mathcal{C})\}_{k\geq 0}$ are called the cohomology of $\mathcal{C}^*$.

By dualizing, to every chain complex $\mathcal{C} = \{C_k, \partial_k\}$ we can associate a canonical cochain complex by $\mathcal{C}^* = \{C^k, \partial_k^*\}$, where $C^k = Hom(C_k, R)$ and ∂_k^* is the dual map defined by $\partial_k^*(a) = a \circ \partial_{k+1}$. We define cochain complex with coefficients in G by $\mathcal{C}_G :=$Hom $(\mathcal{C}, G) = \{Hom_R(C_k, G), \partial_k^*\}$, and define $H^*(C; G) = H^*(\mathcal{C}_G)$.

Given chain complexes $\mathcal{C}' = \{C'_k, \partial'_k\}$ and $\mathcal{C} = \{C_k, \partial_k\}$, a *chain map* $f : \mathcal{C}' \to \mathcal{C}$ is a collection of maps $f = \{f_k : C'_k \to C_k\}$ commuting with boundaries $\partial_k \circ f_k = f_{k-1} \circ \partial'_k$. It follows from this definition that such a map induces a map on the homology groups $f_* : H(\mathcal{C}') \to H_*(\mathcal{C})$. If f is induced by inclusions, we say $\mathcal{C}'$ is a subcomplex of $\mathcal{C}$, and denoted it by $\mathcal{C}' \subset \mathcal{C}$; in this case $\mathcal{C}/\mathcal{C}'$ induces a chain complex structure by the obvious way and then we define the relative homology by $H(\mathcal{C}, \mathcal{C}') = H(\mathcal{C}/\mathcal{C}')$. Similarly, a *cochain map* between two cochains complexes $f : \mathcal{C}^* \to \mathcal{D}^*$ is a collection of maps $\{f_k : C^k \to D^k\}$ commuting with coboundaries, and hence induces map in cohomology $f^* : H^*(\mathcal{C}) \to H^*(\mathcal{D})$. In particular every chain map $f : \mathcal{D} \to \mathcal{C}$ induces a cochain map $\mathcal{C}^* \to \mathcal{D}^*$ by dualizing, and hence it induces map in cohomology $f^* : H^*(\mathcal{C}) \to H^*(\mathcal{D})$.

We say $0 \to \mathcal{C}' \xrightarrow{\alpha} \mathcal{C} \xrightarrow{\beta} \mathcal{C}'' \to 0$ is a short exact sequence of chain complexes if α, β are chain maps, such that for each k the sequence $0 \to C'_k \xrightarrow{\alpha_k} C_k \xrightarrow{\beta_k} C''_k \to 0$ is exact.

Proposition 1.9. *Every short exact sequence of chain complexes* $0 \to \mathcal{C}' \xrightarrow{\alpha} \mathcal{C} \xrightarrow{\beta} \mathcal{C}'' \to 0$ *induces a long exact sequence in their homology groups, for some homomorphism* ∂_*

$$.. \to H_k(\mathcal{C}') \xrightarrow{\alpha_*} H_k(\mathcal{C}) \xrightarrow{\beta_*} H_k(\mathcal{C}'') \xrightarrow{\partial_*} H_{k-1}(\mathcal{C}') \to ..$$

Proof. Once we define ∂_*, checking exactness will be a routine diagram chase. We have following commuting diagrams, where the horizontal sequences are exact.

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & C'_k & \xrightarrow{\alpha_k} & C_k & \xrightarrow{\beta_k} & C''_k & & \longrightarrow 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 \longrightarrow & & C'_{k-1} & \xrightarrow{\alpha_{k-1}} & C_{k-1} & \xrightarrow{\beta_{k-1}} & C''_{k-1} & & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Pick $\{c''\} \in H_k(\mathcal{C}'')$ where $c'' \in Z''_k$ is a cycle representing i.e. $\partial(c'') = 0$, then we can pick $c \in C_k$ with $\beta_k(c) = c''$, hence $\beta_{k-1}\partial(c) = \partial\beta_k(c) = \partial(c'') = 0$, therefore there is $c' \in C'_{k-1}$ with $\alpha_{k-1}(c') = \partial(c)$, then simply define $\partial_*\{c''\} = \{c'\}$ (check c' is cycle by diagram chasing). In short we define $\partial_*\{c''\} = \{\alpha_{k-1}^{-1}\partial_k\beta_k^{-1}(c'')\}$. □

There is also a cohomological version of this Proposition, whose proof is similar, where $\delta^*(c'') = \alpha_{k-1}^{-1}\delta_k\beta_k^{-1}(c'')$ plays the role of ∂_*.

Proposition 1.10. *Any exact sequence of cochain complexes* $0 \to \mathcal{C}^*_{'} \xrightarrow{\alpha} \mathcal{C}^* \xrightarrow{\beta} \mathcal{C}^*_{''} \to 0$ *induces a long exact sequence in their cohomology groups, for some homomorphism* δ^*

$$.. \to H^k(\mathcal{C}^*_{'}) \xrightarrow{\alpha^*} H^k(\mathcal{C}^*) \xrightarrow{\beta^*} H^k(\mathcal{C}^*_{''}) \xrightarrow{\delta^*} H^{k+1}(\mathcal{C}') \to ..$$

An *augmentation* of a chain complex $\mathcal{C} = \{C_k, \partial_k\}_{k \geq 0}$ is an epimorphism $\epsilon : C_0 \to R$ such that $\epsilon \circ \partial_1 = 0$. Homology of the extended chain complex $.. \to C_1 \to C_0 \xrightarrow{\epsilon} R \to 0$ is called the reduced homology and denoted by $\tilde{H}(\mathcal{C})$.

Two chain maps $f, g : \mathcal{C} \to \mathcal{C}'$ are said to be *chain homotopic* if there is a degree 1 map $F : \mathcal{C} \to \mathcal{C}'$, i.e. $F = \{F_k : C_k \to C'_{k+1}\}$, such that $f - g = F_{k-1} \circ \partial_k - \partial'_{k+1} \circ F_k$. Chain homotopic maps induce the same maps in the homology, i.e. $f_* = g_*$.

1.3.1 Tor(A,B)

Let R be a ring with unit and A, B be R-modules. The tensor product $A \otimes_R B$ is defined to be the R-module generated by $\{a \otimes b \mid a \in A, b \in B\}$ with the following relations:

- $(ra + r'a') \otimes b = r(a \otimes b) + r'(a' \otimes b)$
- $a \otimes (rb + r'b') = r(a \otimes b) + r'(a \otimes b')$

Let G be an R module, and $\mathcal{C} = \{C_k, \partial_k\}$ be a chain complex, by tensoring with G we can define a new chain complex $\mathcal{C} \otimes G = \{C_k \otimes_R G, \partial_k \otimes 1\}$ and define homology with coefficients in G by $H_*(\mathcal{C}; G) = H(\mathcal{C} \otimes_R G)$.

Now take the *free resolution of* A, which means the chain complex $\mathcal{C} = \{C_k, \partial_k\}$, is obtained by the following process: Let $C_0 = F(A)$ be the free R-module C_0 generated by the generators of A, and $\epsilon : C_0 \to A$ be the obvious epimorphism, let C_1 be the free R-module generated by the generators of the submodule $ker(\epsilon)$ (which may not be free), and we get an epimorphism $\partial_1 : C_1 \to C_0$, and we continue this process.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\epsilon} & A & \longrightarrow 0 \\ & \shortparallel & & \shortparallel & & \shortparallel & & & \\ & F(ker(\partial_1)) & & F(ker(\epsilon)) & & F(A) & & & \end{array}$$

Definition 1.11. $Tor_k(A, B) := H_k(\mathcal{C} \otimes B)$ and $A * B = Tor_1(A, B)$

In case R is PID, every submodule of a free R-module is free, in the above process we can choose $0 = C_2 = C_3 = ..$; and the free resolution of A becomes $0 \to C_1 \to C_0 \to A \to 0$. From now on we will assume R is PID, such as $R = \mathbb{Z}$ (and $\mathbb{Z}$-modules are abelian groups). Therefore $A * B = ker(\partial_1 \otimes 1)$ which fits into the exact sequence:

$$0 \longrightarrow A * B \longrightarrow C_1 \otimes B \xrightarrow{\partial_1 \otimes 1} C_0 \otimes B \longrightarrow A \otimes B \longrightarrow 0$$

Example 1.1. *If A is free then $A * B = 0$, because the free resolution of A is*

$$0 \to A \to A \to 0$$

Example 1.2. $(R/aR) * B = \{b \in B | ab = 0\}$, *since* R/aR *resolves as*

$$0 \longrightarrow R \xrightarrow{\times a} R \longrightarrow R/aR$$

$$\begin{array}{ccc} R \otimes_R B & \xrightarrow{a \otimes 1} & R \otimes_R B \\ \cong & & \cong \\ B & \xrightarrow{\times a} & B \end{array}$$

Lemma 1.12. *A or B is torsion free then $A * B = 0$.*

Proof. Suppose B is free $B \cong R \oplus \cdots \oplus R$, then take free resolution of A

$$0 \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} A \to 0$$

$$0 \to C_1 \otimes R \xrightarrow{\partial_1} C_0 \otimes R \text{ is monic } \implies A * B = 0 \qquad \square$$

Lemma 1.13. *If $0 \to B' \xrightarrow{\alpha'} B \xrightarrow{\beta'} B'' \to 0$ is exact then $\exists$ an exact sequence*

$$0 \to A * B' \xrightarrow{1 * \alpha'} A * B \xrightarrow{1 * \beta'} A * B'' \to A \otimes B' \xrightarrow{1 \otimes \alpha'} A \otimes B \xrightarrow{1 \otimes \beta'} A \otimes B''$$

Proof. Let $0 \to C_1 \to C_0 \to A$ free resolution of A, let $\mathcal{C} = \{0 \to C_1 \xrightarrow{\partial_1} C_0\}$, then

$$H_k(\mathcal{C}) = \begin{cases} 0 & \text{if } k \neq 0 \\ A & \text{if } k = 0 \end{cases}$$

Since $\mathcal{C}$ is free $0 \to \mathcal{C} \otimes B' \to \mathcal{C} \otimes B \to \mathcal{C} \otimes B'' \to 0$ is exact $\implies \exists$ exact sequence;

$$\begin{array}{ccccccccccc} 0 \longrightarrow H_1(C, B') & \longrightarrow & H_1(C, B) & \longrightarrow & H_1(C, B'') & \to & H_0(C, B') & \longrightarrow & H_0(C, B) & \longrightarrow & H_0(C, B'') \\ \| & & \| & & \| & & \| & & \| & & \| \\ A * B' & & A * B & & A * B'' & & A \otimes B' & & A \otimes B & & A \otimes B'' \end{array}$$

$\square$

Corollary 1.14. $A * B \cong B * A$

Proof. A free resolution $0 \to C_1 \to C_0 \to B \to 0$ of B gives the exact sequence:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A * C_1 & \longrightarrow & A * C_0 & \longrightarrow & A * B & \longrightarrow & A \otimes C_1 & \longrightarrow & A \otimes C_0 & \longrightarrow & A \otimes B & \longrightarrow & 0 \\
 & & & & \shortparallel & & & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
 & & & & 0 & \longrightarrow & B * A & \longrightarrow & C_1 \otimes A & \longrightarrow & C_0 \otimes A & \longrightarrow & B \otimes A & \longrightarrow & 0
\end{array}$$

□

Corollary 1.15. *Let* $TorA \overset{i}{\hookrightarrow} A$, $TorB \overset{j}{\hookrightarrow} B$ *be torsion submodules of* R*-modules then*

$$i * j : TorA * TorB \cong A * B$$

Proof. The sequence $0 \to TorB \to B \to B/TorB \to 0$ and $A * (B/TorB) = 0 \implies$
$i * j : A * TorB \xrightarrow{\cong} A * B$. By a similar argument $i * j : TorA * TorB \xrightarrow{\cong} A * TorB$ □

Theorem 1.16. *Let* C *be a free chain complex over* R *and* G *is an* $R-$*module, then we have the following exact sequence:*

$$0 \longrightarrow H_k(C) \otimes G \longrightarrow H_k(C; G) \longrightarrow H_{k-1}(C) * G \longrightarrow 0$$

Proof. The chain complexes $\mathcal{C} = \{C_k, \partial_k\}$, $\mathcal{B} = \{B_{k-1}, \partial_{k-1}\}$, $\mathcal{Z} = \{Z_k, \partial_k\}$ fits into the following short exact sequence, where α =inclusion and $\beta = \partial$.

$$0 \to Z \xrightarrow{\alpha} C \xrightarrow{\beta} B \to 0$$

Since each term of these chain complexes is free, tensoring them with G keeps it exact.

$$0 \to Z \otimes G \to C \otimes G \to B \otimes G \to 0$$

This induces the following long exact sequence in homologies:

$$
\begin{tikzcd}
\cdots \arrow[r] & H_{k+1}(B:G) \arrow[r, "\partial_*"] & H_k(Z:G) \arrow[r] & H_k(C:G) \arrow[r] & H_k(B:G) \arrow[r, "\partial_*"] & H_{k-1}(Z:G) \\
& B_k \otimes G \arrow[u, phantom, "\cong"] \arrow[r, "\alpha_k \otimes 1"] & Z_k \otimes G \arrow[u, phantom, "\cong"] & & B_{k-1} \otimes G \arrow[u, phantom, "\cong"] \arrow[r, "\alpha_{k-1} \otimes 1"] & Z_{k-1} \otimes G \arrow[u, phantom, "\cong"]
\end{tikzcd}
$$

From this we induce the following short exact sequence:

$$0 \to \operatorname{coker}(\alpha_k \otimes 1) \to H_k(C:G) \to ker(\alpha_{k-1} \otimes 1) \to 0$$

Now consider the following exact sequence, which is a free resolution of H_k

$$0 \to B_k \to Z_k \to \mathbb{H}_k \to 0$$

From definition, we get the following exact sequence, which implies the desired result.

$$0 \to H_k * G \to B_k \otimes G \xrightarrow{\alpha_k \otimes 1} Z_k \otimes G \to H_k \otimes G \to 0$$

$$ker(\alpha_k \otimes 1) \cong H_k * G \;\text{ and }\; \operatorname{coker}(\alpha_k \otimes 1) \cong H_k \otimes G$$

□

1.3.2 Ext(A,B)

Let A and B be R-modules and $0 \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} A \to 0$ be the free resolution of A.

Definition 1.17. $\operatorname{Ext}(A,B) := H^1(\mathcal{C}; B)$, where $\mathcal{C} = \{C_k, \partial_k\}_{k=0,1}$

Therefore $\operatorname{Ext}(A,B)$ fits into the following exact sequence as the cokernel of ∂_1^*

$$0 \to Hom(A,B) \xrightarrow{\epsilon^*} Hom(C_0,B) \xrightarrow{\partial_1^*} Hom(C_1,B) \to Ext(A,B) \to 0$$

Example 1.3. *$Ext\,(R/rR, B) \cong B/rB$. To see this we take the free resolution*

$$0 \to R \xrightarrow{r} R \to R/rR \to 0$$

$$\begin{array}{ccccccc} 0 \longrightarrow Hom(R/rR) \longrightarrow & Hom(R,B) & \xrightarrow{r^*} & Hom(R,B) & & \\ & \wr\| & & \wr\| & & \\ & R\otimes B & \xrightarrow{\quad r^* \quad} & R\otimes B & & \\ & \wr\| & & \wr\| & & \\ & B & \xrightarrow{\quad \times r \quad} & B & \quad \Rightarrow & coker(r^*) = B/rB \end{array}$$

So in particular we have $Ext(\mathbb{Z}_n, \mathbb{Z}_m) = Ext(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_d$ *where* $d = gcd(n,m)$

Theorem 1.18. *Let* $\mathcal{C}$ *be a free chain complex over* R*,* G *is an* R*-module, then we have the following short exact sequence.*

$$0 \longrightarrow Ext(H_{k-1}(\mathcal{C}), G) \to H^k(\mathcal{C} : G) \to Hom(H_k(\mathcal{C}), G) \longrightarrow 0$$

Proof. From the usual exact sequence of the chain complexes $0 \to \mathcal{Z} \xrightarrow{\alpha} \mathcal{C} \xrightarrow{\beta} \mathcal{B} \to 0$, where $\mathcal{C} = \{C_k, \partial_k\}$, $\mathcal{B} = \{B_{k-1}, \partial_{k-1}\}$, $\mathcal{Z} = \{Z_k, \partial_k\}$, we get the short exact sequence:

$0 \to Hom(\mathcal{B}, G) \xrightarrow{\beta^*} Hom(\mathcal{C} : G) \xrightarrow{\alpha^*} Hom(\mathcal{Z} : G) \to 0$ implying the long exact sequence :

$$Hom(Z_{k-1}, G) \xrightarrow{\alpha^*_{k-1}} Hom(B_{k-1}, G) \longrightarrow H^k(C : G) \longrightarrow Hom(Z_k, G) \xrightarrow{\alpha^*_k} Hom(B_k, G)$$

So the following is exact: $0 \to \mathrm{coker}(\alpha^*_{k-1}) \to H^k(C : G) \to ker(\alpha^*_k) \to 0$

On the other hand, the resolution $0 \to B_k \to Z_k \to H_k \to 0$ gives the exact sequence

$$0 \longrightarrow Hom(H_k, G) \longrightarrow Hom(Z_k, G) \xrightarrow{\alpha^*_k} Hom(B_k, G) \longrightarrow Ext(H_k, G) \longrightarrow 0$$

$$\Longrightarrow \quad \begin{array}{rcl} ker(\alpha^*_k) & \cong & Hom(H_k, G) \\ \mathrm{coker}(\alpha^*_k) & \cong & Ext(H_k, G) \end{array}$$

□

1.3.3 Künneth Formula

Let $C = \{C_k, \partial_k\}$, $C' = \{C'_k, \partial'_k\}$ be chain complexes, we define the following,

$$\begin{aligned} C \otimes C' &= \{(C \otimes C')_q, \partial_q\} \\ C * C' &= \{(C * C')_q, \partial_q\} \end{aligned}$$

$$\begin{aligned} (C \otimes C')_q &= \bigoplus_{i+j=q} C_i \otimes C'_j \\ (C * C')_q &= \bigoplus_{i+j=q} C_i * C'_j \end{aligned}$$

$$\begin{aligned} \partial(c_i \otimes c_j) &= \partial c_i \otimes c_j + (-1)^i c_i \otimes \partial c_j \\ \partial(c_i * c_j) &= \partial c_i * c_j + (-1)^i c_i * \partial c_j. \end{aligned}$$

If C, C' are augmented chain complexes, so is $C \otimes C'$, with the augmentation

$$C_0 \otimes C'_0 \xrightarrow{\varepsilon \otimes \varepsilon'} R \otimes_R R \cong R$$

Theorem 1.19. *Let C, C' be chain complexes, Then, there is a short exact sequence:*

$$0 \to [H(C) \otimes H(C')]_q \to H_q(C \otimes C') \to [H(C) * H(C')]_{q-1} \to 0$$

Proof. Let $Z' = \{Z_q(C')\}$ and $B' = \{B_{q-1}(C')\}$. Tensoring the exact sequence

$$0 \longrightarrow Z' \xrightarrow{i} C' \xrightarrow{\partial} B' \longrightarrow 0$$

gives the following short exact sequence

$$0 \longrightarrow C \otimes Z' \xrightarrow{\alpha} C \otimes C' \xrightarrow{\beta} C \otimes B' \longrightarrow 0$$

where $\alpha = Id \otimes i$ and $\beta = Id \otimes \partial$, which implies we have a long exact sequence

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[r, "\partial_*"] & \mathrm{H}_q(C\otimes Z') \arrow[r] & \mathrm{H}_q(C\otimes C') \arrow[r] \arrow[dr] & \mathrm{H}_q(C\otimes B') \arrow[r, "\partial_*"] & \mathrm{H}_{q-1}(C\otimes Z') \arrow[r] & \cdots \\
& \mathrm{coker}\partial_* \arrow[ur] & & \ker\partial_* \arrow[dr] & & \\
0 \arrow[ur] & & & & 0 &
\end{tikzcd}
$$

$$
\begin{aligned}
(C\otimes Z')_q &= \oplus_j\ C_{q-j}\otimes Z_j(C') \\
(C\otimes B')_q &= \oplus_j\ C_{q-j}\otimes B_{j-1}(C')
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{H}_q\left(C\otimes Z'\right) &= \oplus_{i+j=q}\ \mathrm{H}_i(C)\otimes Z_j(C') \\
\mathrm{H}_q\left(C\otimes B'\right) &= \oplus_{i+j=q-1}\ \mathrm{H}_i(C)\otimes B_{j-1}(C')
\end{aligned}
$$

$$
\begin{aligned}
\partial_*(c\otimes\partial c') &= \alpha^{-1}\partial\beta^{-1}(c\otimes\partial c') \\
&= \alpha^{-1}\partial(c\otimes c') = \alpha^{-1}(\partial c\otimes c' + (-1)^i c\otimes\partial c') \\
&= (-1)^i\mathbb{1}\otimes\gamma_j(c\otimes\partial c')
\end{aligned}
$$

where $\gamma_j : B_j(C') \hookrightarrow Z_j(C')$ is the inclusion.

$$0\to\oplus_{i+j=q}\mathrm{coker}\left((-1)^i\otimes\gamma_j\right)\to\mathrm{H}_q(C\otimes C')\to\oplus_{i+j=q-1}\ker\left((-1)^i\otimes\gamma_j\right)\to 0$$

Consider

$$0\to B_j(C')\xrightarrow{(-1)^i\gamma_j} Z_j(C')\to\mathrm{H}_j(C')\to 0$$

which is the free resolution of $H_j(C')$ implying the exact sequence:

$$0\to\mathrm{H}_i(C)*\mathrm{H}_j(C')\to\mathrm{H}_i(C)\otimes B_j(C')\to\mathrm{H}_i(C)\otimes Z_j(C')\to\mathrm{H}_i(C)\otimes\mathrm{H}_j(C')\to 0$$

$$
\begin{aligned}
\implies\quad \bigoplus_{i+j=q}\mathrm{coker}[(-1)^i\otimes\gamma_j] &= \bigoplus_{i+j=q}\mathrm{H}_i(C)\otimes\mathrm{H}_j(C') \\
\bigoplus_{i+j=q-1}\ker[(-1)^i\otimes\gamma_j] &= \bigoplus_{i+j=q-1}\mathrm{H}_i(C)*\mathrm{H}_j(C') \qquad \square
\end{aligned}
$$

1.4 Homology and cohomology of topological spaces

1.4.1 Singular homology

To every topological space X we can associate the chain complex $\mathcal{S}(X) = \{S_k(X), \partial_k\}_{k\geq 0}$, where $S_k(X)$ is the infinite dimensional R-vector space generated by continuous maps $f : \Delta_k \to X$, and $\partial_k : S_k(X) \to S_{k-1}(X)$ is the linear map defined on generators by

$$\partial_k(f) = \sum_{i=0}^{k} (-1)^i f|_{\Delta_K^{(i)}}$$

Here Δ_k is the standard k-simplex $[w_0, w_1, .., w_n]$, and $\Delta_k^{(i)} = [w_0, ..\hat{w}_i, ..w_k]$ is the i'th face of Δ_k. It follows from definitions that $\partial_{k-1} \circ \partial_k = 0$. The homology groups corresponding to this chain complex is called the *singular homology groups* of X and denoted by $H_*(X)$. It follows from definition that any continuous map between spaces $f : X \to Y$ induces homomorphism on the homology groups $f_* : H_*(X) \to H_*(Y)$, which is functorial with respect to composition: $(f \circ g)_* = f_* \circ g_*$. Hence the map $X \mapsto H_*(X)$ is a functor from the category of topological spaces and continuous maps, to the category of groups and homomorphisms, in particular $H_*(X)$ only depends on the homeomorphism type of X. We can canonically augment the chain complex $\mathcal{S}(X)$ by extending with the map $\epsilon : S_0(X) \to R$ by defining $\epsilon(f) = 1$ on generators $f : \Delta_o \to X$. We denote the homology of this augmented chain complex *reduced homology* and denote it by $\tilde{H}(X; R)$.

It follows that if P is a point then $H_k(P) = 0$ if $k > 0$ and $H_0(P) = R$. Also if $Z = X \sqcup Y$ (disjoint union) $\implies$ $H_*(Z) = H_*(X) \oplus H_*(Y)$.

The inclusion $A \overset{i}{\hookrightarrow} X$ induce inclusion $i_* : \mathcal{S}(A) \hookrightarrow \mathcal{S}(X)$, then we can define $H_*(X, A) = H_*(\mathcal{S}(X)/\mathcal{S}(A))$. By Proposition 1.9 we see that the short exact sequence $0 \to \mathcal{S}(A) \to \mathcal{S}(X) \to \mathcal{S}(X)/\mathcal{S}(A) \to 0$ implies the following long exact sequence:

$$.. \to H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X, A) \xrightarrow{\partial_*} H_{k-1}(A) \to .. \tag{1.1}$$

where $(X, \phi) \overset{j}{\hookrightarrow} (X, A)$ is the obvious inclusion. Also homology groups satisfy the following basic properties:

(1) (**Homotopy invariance**) Homotopic maps $f, g : (X, A) \to (Y, B)$ induce the same homomorphisms on the homology, i.e $f \simeq g \implies f_* = g_* : H_*(X, A) \to H_*(Y, B)$

(2) (**Excision**) Given $U \subset \bar{U} \subset A \subset X$ with U open, then the inclusion mapping $(X - U, A - U) \hookrightarrow (X, A)$ induces isomorphism $H_*(X - U, A - U) \cong H_*(X, A)$.

From (1) we see that the homology of any contractible space is the same as the homology of a point. Using this fact and (1.1) we get

$$H_k(D^p, S^{p-1}) = \begin{cases} 0 & \text{if } k \neq p \\ R & \text{if } k = p \end{cases} \tag{1.2}$$

For CW-complexes computing this homology groups are somewhat easier. For example when (K, L) is a CW-pair by using the above properties and Proposition(1.8) we can calculate: If there is $L \subset U \subset K$ which U deformation retracts to L then

$$H_q(K, L) \cong H_q(K, U) \cong H_q(K - L, U - L) \tag{1.3}$$

$$H_q(K/L) = H_q(K/L, *) = H_q(K \cup c(L), *) = H_q(K \cup c(L), c(L)) \cong H_q(K, L) \tag{1.4}$$

Therefore the usual long exact sequence of the pair (K, L) becomes:

$$.. \to H_q(L) \to H_q(K) \to H_q(K/L) \xrightarrow{\partial_*} H_{q-1}(L) \to .. \tag{1.5}$$

So the inclusions of CW-complexes $N \subset L \subset K$ give rise to the long exact sequence:

$$.. \to H_q(L/N) \to H_q(K/N) \to H_q(K/L) \xrightarrow{\partial_*} H_{q-1}(L/N) \to .. \tag{1.6}$$

Just using (1.2) we can calculate homology groups of CW-complexes from their attaching maps. For this we need to equate the singular homology with CW-homology, which will be defined in the next section.

1.4.2 CW-homology

Let X be a finite CW-complex $X = \bigsqcup X^p$, where X^p be its p^{th} skeleton, that is $X^p = X^{p-1} + \sum_{i=1}^{n_p} e_i^p$, where $\{e^p\}$ are $p-$cells. For simplicity lets take $R = \mathbb{Z}$. Then from above:

$$H_k(X^p, X^{p-1}) = \begin{cases} 0 & \text{if } k \neq p \\ \mathbb{Z}^{n_p} & \text{if } k = p \end{cases} \tag{1.7}$$

CW- chain complex of X is $\mathcal{C}(X) = \{\mathcal{C}_p(X), \bar{\partial}_p\}$, where $\mathcal{C}_p(X) = H_p(X^p, X^{p-1})$ (free abelian group generated by p-cells of X), and $\bar{\partial}_p = j_* \circ \partial_p$ as defined by the following diagram (clearly from diagram chasing we have $\overline{\partial}^2 = 0$):

$$\begin{tikzcd}
 & 0 \arrow[d, "j_*"] & & \\
H_{p+1}(X^{p+1}, X^p) \arrow[r, "\partial_{p+1}"] \arrow[d, equal] \arrow[dr, dashed, "\overline{\partial}_{p+1}"'] & H_p(X^p) \arrow[d, "j_*"] \arrow[dr, dashed] & & \\
\mathcal{C}_{p+1}(X) & H_p(X^p, X^{p-1}) \arrow[r, "\partial_p"] \arrow[d, equal] \arrow[dr, dashed, "\overline{\partial}_p"'] & H_{p-1}(X^{p-1}) \arrow[d, "j_*"] & \\
 & \mathcal{C}_p(X) & H_{p-1}(X^{p-1}, X^{p-2}) \arrow[r, phantom, "="] & \mathcal{C}_{p-1}(X)
\end{tikzcd}$$

Theorem 1.20. $H_p(X) \cong H_p(\mathcal{C}(X))$

Proof. From $X^{p-2} \subset X^p \subset X^{p+1}$ and $X^{p-2} \subset X^{p-1} \subset X^p$ we get exact sequences:

$$\begin{tikzcd}
\mathcal{C}_{p+1}(X) \arrow[d, equal] & H_p(X^{p-1}, X^{p-2}) = 0 \arrow[d] & & \\
H_{p+1}(X^{p+1}, X^p) \arrow[r, "\partial_{p+1}"] \arrow[d, "\bar{\partial}_{p+1}"] & H_p(X^p, X^{p-2}) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_p(X^{p+1}, X^{p-2}) \arrow[r] & H_p(X^{p+1}, X^p) \arrow[d, equal] \\
\mathcal{C}_p(X) \arrow[r, phantom, "="] \arrow[d, "\bar{\partial}_p"] & H_p(X^p, X^{p-1}) \arrow[d, "\bar{\partial}_p"] & & 0 \\
\mathcal{C}_{p-1}(X) \arrow[r, phantom, "="] & H_{p-1}(X^{p-1}, X^{p-2}) & &
\end{tikzcd}$$

- Second vertical row $\implies ker\bar{\partial}_p \cong imj_* \cong H_p(X^p, X^{p-2})$
- $\bar{\partial}_{p+1} = j_* \circ \partial_{p+1}$ and j_* is 1-1 $\implies im\bar{\partial}_{p+1} \cong im\partial_{p+1} = ker(i_*) \implies$

$$\begin{aligned} H_p(\mathcal{C}(X)) &= ker\bar{\partial}_p/im\bar{\partial}_{p+1} \cong H_p(X^p, X^{p-2})/ker(i_*) \\ &= H_p(X^{p+1}, X^{P-2}) \cong H_p(X^{p+1}) \cong H_p(X) \end{aligned}$$

□

1.4.3 Čech cohomology

A *presheaf* of abelian groups on a topological space X is an assignments of its open sets to abelian groups $U \mapsto \mathcal{F}(U)$ satisfying some compatibility conditions: For $V \subset U$ there are so called "restriction" maps $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$, such that $\rho_U^U = id$, and when $W \subset V \subset U \implies \rho_W^V \circ \rho_V^U = \rho_W^U$. We will abbreviate $\rho_V^U(s) = \rho(s) = s|$

Presheaf is called a *sheaf* if any open cover $\mathcal{U} = \{U_\alpha\}$ of X we have

1. If $s, t \in \mathcal{F}(X)$ and $s\big|_{U_\alpha} = s\big|_{U_\beta}$ for all α, β then $s = t$.
2. Given $s_\alpha \in \mathcal{F}(U_\alpha)$ with $s_\alpha\big|_{U_\alpha \cap U_\beta} = s_\beta\big|_{U_\alpha \cap U_\beta}$ for all α, β, then there is $s \in \mathcal{F}(X)$ with $s\big|_{U_\alpha} = s_\alpha$.

To any open covering $\mathcal{U} = \{U_\alpha\}$ of X we associate a simplicial complex $N(\mathcal{U})$, called the *nerve* of $\mathcal{U}$, where p-simplexes are $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. A p-cochain $\sigma = \{\sigma_{\alpha_0,\ldots,\alpha_p}\}$ is an assignment $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \mapsto \sigma_{\alpha_0,\ldots,\alpha_p} \in \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$ which is skew-symmetric in the indices $\sigma_{\alpha_0,\ldots,\alpha_i,\alpha_{i+1},\ldots,\alpha_p} = -\sigma_{\alpha_0,\ldots,\alpha_{i+1},\alpha_i,\ldots,\alpha_p}$. Let $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ be the group generated by all p-cochains, we can identify:

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$$

We define coboundary maps $\delta : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$ by:

$$(\delta\sigma)_{\alpha_0,\ldots,\alpha_{p+1}} = \sum (-1)^i \rho(\sigma_{\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\alpha_{p+1}}) \ \text{, where}$$

$$\rho = \rho^{U_{\alpha_0}\cap\cdots\cap\hat{U}_{\alpha_i}\cap\cdots\cap U_{\alpha_{p+1}}}_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_{p+1}}}$$

In short we will write $(\delta\sigma)_{\alpha_0\ldots\alpha_{p+1}} = \sum(-1)^i \sigma_{\alpha_0\ldots\widehat{\alpha}_i\ldots\alpha_{p+1}}$, then we define the groups

$$\begin{aligned}
\mathcal{Z}^p(\mathcal{U},\mathcal{F}) &= ker(\delta) = \text{ cocycles} \\
\mathcal{B}^p(\mathcal{U},\mathcal{F}) &= im(\delta) = \text{ coboundaries} \\
\mathrm{H}^p(\mathcal{U},\mathcal{F}) &= \mathcal{Z}^p(\mathcal{U},\mathcal{F})/\mathcal{B}^p(\mathcal{U},\mathcal{F})
\end{aligned}$$

Remark 1.21. Sheaves satisfy $\mathrm{H}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(X)$, since $(\delta\sigma)_{\alpha\beta} = \sigma_\alpha - \sigma_\beta$. We can define sheafs of nonabelian Lie groups, but in that case only the first sheaf cohomology can be defined, for example $\mathrm{H}^1(X, GL(n,\mathbb{C}))$ is identified by the vector bundles over X.

If $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}, \mathcal{V} = \{V_\beta\}_{\beta\in J}$ are two coverings of X we say $\mathcal{U}$ is finer then $\mathcal{V}$ and denote by $\mathcal{U} > \mathcal{V}$, if there is a map $\phi : I \to J$ with $U_\alpha \subset V_{\phi(\alpha)}$. ϕ is called a refinement map, it induces a chain map:

$$\begin{aligned}
&\phi^\# : \mathcal{C}^p(\mathcal{V},\mathcal{F}) \to \mathcal{C}^p(\mathcal{U},\mathcal{F}) \text{ is a chain map defined by} \\
&(\phi^\#\sigma)_{\alpha_0,\ldots,\alpha_p} = \rho(\sigma_{\phi(\alpha_0),\ldots,\phi(\alpha_p)})
\end{aligned}$$

ρ is the obvious restriction map. $\phi^\#$ is a chain map because:

$$\begin{aligned}
(\delta(\phi^\#\sigma))_{\beta_0,\ldots,\beta_{q+1}} &= \sum(-1)^i(\phi^\#\sigma)_{\beta_0,\ldots,\widehat{\beta}_i,\ldots,\beta_{q+1}} \\
&= \sum(-1)^i\sigma_{\phi(\beta_0),\ldots,\widehat{\phi(\beta_i)},\ldots,\phi(\beta_{q+1})} \\
&= (\delta\sigma)_{\phi(\beta_0),\ldots,\phi(\beta_{q+1})} \\
&= (\phi^\#\delta\sigma)_{\beta_0,\ldots,\beta_{q+1}}
\end{aligned}$$

Lemma 1.22. *If $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}, \mathcal{V} = \{V_\beta\}_{\beta\in J}$ and $\phi, \psi : J \to I$ are two refinement maps, then $\phi^\#$ and $\psi^\#$ are chain homotopic, hence they induce the same map on cohomology.*

Proof. Define $K : \mathcal{C}^q(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{q-1}(\mathcal{V}, \mathcal{F})$ by

$$(K\sigma)_{\beta_0,\ldots,\beta_{q-1}} = \sum (-1)^i \sigma_{\phi(\beta_0),\ldots,\phi(\beta_i),\psi(\beta_i),\ldots,\psi(\beta_{q-1})}.$$

Then it is easy to check that $\psi^\# - \phi^\# = \delta K + K\delta$. $\square$

Recall that given abelian groups $\{G_i\}$ if for all $a > b$, there is $f_b^a : G_b \to G_a$ with

1. $f_a^a = Id$
2. $f_c^a = f_c^b \circ f_b^a \quad \text{if} \quad c < b < a$

we can define $\varinjlim G_\alpha = \sqcup G_\alpha / \sim$ (direct limit), where $\sim$ is defined by

$$\text{for } g \in G_b,\ h \in G_c,\ g \sim h \text{ if } f_b^a(g) = f_c^a(h)$$

Definition 1.23. $\mathrm{H}^*(X, \mathcal{F}) = \varinjlim \mathrm{H}^*(\mathcal{U}, \mathcal{F})$

So to compute $\mathrm{H}^*(X, \mathcal{F})$ we choose a fine enough cover $\mathcal{U} = \{U_\alpha\}$ then compute $\mathrm{H}^*(\mathcal{U}, \mathcal{F})$. We call $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \to 0$ an exact sequence of sheaves, if for some fine open covering $\mathcal{U}$ of X, for each $U \in \mathcal{U}$ there is an exact sequence of abelian groups

$$0 \to \mathcal{F}'(U) \xrightarrow{\alpha} \mathcal{F}(U) \xrightarrow{\beta} \mathcal{F}''(U) \to 0$$

These maps gives rise to the exact sequence of cochain complexes

$$0 \to \mathcal{C}(X, \mathcal{F}') \to \mathcal{C}(X, \mathcal{F}) \to \mathcal{C}(X, \mathcal{F}'') \to 0$$

Proposition 1.10 this implies the following long exact sequence

$$\cdots \to \mathrm{H}^(X, \mathcal{F}') \to \mathrm{H}^p(X, \mathcal{F}) \to \mathrm{H}^p(X, \mathcal{F}'') \xrightarrow{\delta^*} \mathrm{H}^{p+1}(X, \mathcal{F}') \to \ldots$$

For example, to define the δ^* we first pick $\sigma'' \in \mathcal{Z}^p(\mathcal{U}, \mathcal{F}'')$ such that $\sigma'' = \beta(\sigma)$ for some σ, then since $0 = \delta\sigma'' = \delta\beta(\sigma) = \beta(\delta\sigma) \implies \delta\sigma = \sigma'$ for some σ', then let $\delta^*(\sigma'') = \sigma'$.

Remark 1.24. Let M be a complex manifold, let $\underline{\mathbb{Z}}$, $\mathcal{O}$ and $\mathcal{O}^*$ be the sheaf of integers, holomorphic functions and non-vanishing holomorphic functions, respectively; then there is the sheaf exact sequence $0 \to \underline{\mathbb{Z}} \to \mathcal{O} \overset{e^{2\pi i \cdot}}{\to} \mathcal{O}^* \to 0$ (the last map is $f \mapsto e^{2\pi i f}$)

Remark 1.25. Let $\mathcal{U} = \{U_\alpha\}$ an open covering of a smooth manifold M, such that all nonempty intersections $U_{\alpha_0} \cap .. \cap U_{\alpha_p}$ are contractible. Pick a closed form $\omega \in \Omega^p(M)$, $d\omega = 0$. Then, by Poincaré Lemma, $\omega\big|_{U_\alpha} = d\omega_\alpha$ for some $\omega_\alpha \in \Omega^{p-1}(U_\alpha)$. Thus, $d(\omega_\alpha - \omega_\beta) = 0$ on $U_\alpha \cap U_\beta$. So, $\omega_\alpha - \omega_\beta = d\omega_{\alpha\beta}$ for some $\omega_{\alpha\beta} \in \Omega^{p-1}(U_\alpha \cap U_\beta)$. This, in turn, gives us $d(\omega_{\beta\gamma} - \omega_{\alpha\gamma} + \omega_{\alpha\beta}) = 0$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Again by Poincaré Lemma $\omega_{\alpha\beta} + \omega_{\beta\gamma} + \omega_{\gamma\alpha} = d\omega_{\alpha\beta\gamma}$ for some $\omega_{\alpha\beta\gamma} \in \Omega^{p-2}(U_\alpha \cap U_\beta \cap U_\gamma)$. By continuing this way we get $\omega_{\alpha_0,\dots,\alpha_p} \in \Omega^0(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$. Let $\theta_{\alpha_0,\dots,\alpha_p} = \sum(-1)^i \omega_{\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_p}$. Then, $d\theta_{\alpha_0,\dots,\alpha_p} = 0$ which implies $\left\{\theta_{\alpha_0,\dots,\alpha_p}\right\}$ is a Čech cocycle. This gives us a map $\mathrm{H}^p_{DR}(M) \to \mathrm{H}^*(M, \underline{\mathbb{R}})$, which turns out to be isomorphism. This is the content of the Theorem 1.28

A Sheaf $\mathcal{F}$ on a smooth manifold M is called *fine sheaf*, if multiplying elements of $\mathcal{F}(U)$ by smooth functions keep them in $\mathcal{F}(U)$. So in a fine sheaf elements of $\mathcal{F}(U)$ can be added together with partition of unity functions. The sheaf of smooth differential p-forms Ω^p on M is an example of a fine sheaf.

Proposition 1.26. *If $\mathcal{F}$ is a fine sheaf, then $H^k(X, \mathcal{F}) = 0$ for all $k > 0$.*

Corollary 1.27. *$H^k(M, \Omega^p) = 0$ for all $k > 0$.*

Proof. Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M as in Remark 1.25. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\mathcal{U}$, and

$$\sigma = \{\sigma_{\alpha_0,\dots,\alpha_k}\} \in \mathrm{H}^k(M, \mathcal{F})$$

where $\sigma_{\alpha_0,\dots,\alpha_k}$ is a p-cochain on $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. Let $\tau = \left\{\tau_{\alpha_0,\dots,\alpha_{k-1}}\right\}$ be

$$\tau_{\alpha_0,\dots,\alpha_{k-1}} = \sum \rho_\alpha \sigma_{\alpha,\alpha_0,\dots,\alpha_{k-1}}$$

$$
\begin{aligned}
\Longrightarrow \quad (\delta\tau)_{\alpha_0,\dots\alpha_k} &= \sum_r (-1)^r \tau_{\alpha,\dots\widehat{\alpha_r},\dots,\alpha_k} \\
&= \sum_r (-1)^r \sum_\alpha \rho_\alpha \sigma_{\alpha,\alpha_0,\dots,\widehat{\alpha_r},\dots,\alpha_k}
\end{aligned}
$$

$$
\text{Since } \; 0 = (\delta\sigma)_{\alpha,\alpha_0,\dots,\alpha_k} = \sigma_{\alpha_0,\dots,\alpha_k} - \sum_r (-1)^r \sigma_{\alpha,\alpha_0,\dots,\widehat{\alpha_r},\dots,\alpha_k} \Longrightarrow
$$

$$
\begin{aligned}
(\delta\tau)_{\alpha_0,\dots,\alpha_k} &= \sum_\alpha \rho_\alpha \sum (-1)^r \sigma_{\alpha,\dots,\widehat{\alpha_r},\dots,\alpha_k} \\
&= \sum \rho_\alpha \sigma_{\alpha_0,\dots,\alpha_k} = \sigma_{\alpha_0,\dots,\alpha_k} \qquad \square
\end{aligned}
$$

Theorem 1.28. *(De Rham) If M is a smooth manifold then $H^p(M,\Omega) \cong H^p(M,\mathbb{R})$*

Proof. From the sequence $0 \to \underline{\mathbb{R}} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \to 0$ we get short exact sequences:

$$
\begin{gathered}
0 \to \underline{\mathbb{R}} \to \Omega^0 \xrightarrow{d} \mathcal{Z}^1 \to 0 \\
0 \to \mathcal{Z}^1 \to \Omega^1 \xrightarrow{d} \mathcal{Z}^2 \to 0 \\
\vdots \\
0 \to \mathcal{Z}^{q-1} \to \Omega^{q-1} \xrightarrow{d} \mathcal{Z}^q \to 0
\end{gathered}
$$

Each short exact sequence gives a long exact sequence in homology, by using Corollary 1.27 we get

$$
\begin{aligned}
\mathrm{H}^q(\underline{\mathbb{R}}) &\cong \mathrm{H}^{q-1}(\mathcal{Z}^1) \\
&\cong \mathrm{H}^{q-2}(\mathcal{Z}^2) \\
&\;\vdots \\
&\cong \mathrm{H}^1(\mathcal{Z}^{q-1}) = \mathrm{H}^0(\mathcal{Z}^q)/\mathrm{im}(d) \cong H^q(M,\Omega)
\end{aligned}
$$

Last equality follows from the exact sequence $\mathrm{H}^0(\Omega^q) \xrightarrow{d} \mathrm{H}^0(\mathcal{Z}^q) \twoheadrightarrow H^1(\mathcal{Z}^{q-1}) \to 0$.

$\square$

Chapter 2

Products in cohomology

2.1 Acyclic model theorem

Let $\mathcal{A}$ be a category, and $\mathcal{M} \subset \text{objects}(\mathcal{A})$. We say $(\mathcal{A}, \mathcal{M})$ is a *category with models.*

Let $\mathfrak{A}$ be a category of R-modules, and $G : \mathcal{A} \to \mathfrak{A}$ is a functor. We say G has a *basis* relative to $\mathcal{M}$ if there is $\{d_j \in G(M_j)\}_{j \in J}$ where $M_j \in \mathcal{M}$, such that for every object X of $\mathcal{A}$, $G(X)$ is the free R-module spanned by

$$\{G(u)(d_j)\}_{j \in J, u \in \mathrm{Hom}(M_j, X)} .$$

If $\mathcal{C}$ is the category of augmented chain complexes and $F : \mathcal{A} \to \mathcal{C}$ is a functor $F(X) = \{F_k(X), \partial_k\}$ we say F is *free* (over $\mathcal{M}$) if each $F_k : \mathcal{A} \to \mathfrak{A}$ has a basis. We say F is *acyclic* (over $\mathcal{M}$) if $\widetilde{H}_*(F(M)) = 0$ for all $*$ and $M \in \mathcal{M}$.

Example 2.1. *Let $\mathcal{A}$ be the category of topological spaces and continuous maps, $\mathcal{M} = \{\Delta_k : k \geq 0\}$ and $S : \mathcal{A} \to \mathcal{C}$ be the singular homology functor then S is free and acyclic over $\mathcal{M}$. If $\delta_j : \Delta_j \to \Delta_j$ is the identity map then $\{\delta_j \in S_j(\Delta_j)\}$ is a basis for S_j.*

Theorem 2.1. *Let $(\mathcal{A}, \mathcal{M})$ be a category with models $\mathcal{C}$ be a category of augmented chain complexes $F, F' : \mathcal{A} \to \mathcal{C}$ be functors with F free and F' acyclic. Then, there exists a morphism $\Phi : F \to F'$. If $\Psi : F \to F'$ is any other morphism then Φ, Ψ are chain homotopic.*

Proof. For every $X \in \text{object}(\mathcal{A})$ we must define $\Phi(X) : F(X) \to F'(X)$ such that if $f : Y \to X$ is any map then

$$\begin{tikzcd}
F(Y) \arrow[r, "\Phi(Y)"] \arrow[d, "F(f)"'] & F'(Y) \arrow[d, "F'(f)"] \\
F(X) \arrow[r, "\Phi(X)"'] & F'(X)
\end{tikzcd}$$

commutes. More specifically the following commutes

$$\begin{tikzcd}
{} \arrow[dr, dashed] & & & {} \arrow[dr, dashed] & & & \\
& F_k(Y) \arrow[rrr, "\Phi_k(Y)"] \arrow[dr, "\partial"] \arrow[dd, "F_k(f)"'] & & & F'_k(Y) \arrow[dr, "\partial'"] \arrow[dd, "F'_k(f)"] & & \\
{} \arrow[dr, dashed] & & F_{k-1}(Y) \arrow[rrr, "\Phi_{k-1}(Y)"] \arrow[dd, "F_{k-1}(f)"'] \arrow[dr, dashed] & {} \arrow[dr, dashed] & & F'_{k-1}(Y) \arrow[dd, "F'_{k-1}(f)"] \arrow[dr, dashed] & \\
& F_k(X) \arrow[rrr, "\Phi_k(X)"'] \arrow[dr, "\partial"] & & {} & F'_k(X) \arrow[dr, "\partial'"] & & {} \\
& & F_{k-1}(X) \arrow[rrr, "\Phi_{k-1}(X)"'] \arrow[dr, dashed] & & & F'_{k-1}(X) \arrow[dr, dashed] & \\
& & & {} & & & {}
\end{tikzcd}$$

For each k, let $\left\{d_{k_j} \in F_k(M_j)\right\}$ be a basis for F_k, i.e. $F_k(X)$ is generated by $\{F_k(f)(d_{k_j}) : f \in \text{Hom}(M_j, X)\}$. By taking $Y = M_j$ the above diagram gives

$$\Phi_k(X)F_k(f)(d_{k_j}) = F'_k(f)\Phi_k(M_j)(d_{k_j})$$

so it suffices to define $\Phi_k(M_j)(d_{k_j})$. We will define it by induction. We need to define the dotted arrows

$$\begin{tikzcd} F_1(M_j) \arrow[r, "\partial_1"] \arrow[d, dashed] & F_0(M_j) \arrow[r, two heads, "\varepsilon"] \arrow[d, dashed] & R \\ F_1'(M_j) \arrow[r, "\partial_1'"] & F_0'(M_j) \arrow[ur, "\varepsilon'"] & \end{tikzcd}$$

Since $F'(M_j)$ is acyclic, $\ker(\varepsilon') = \operatorname{im}(\partial_1')$. Hence, ε' induces an isomorphism $\varepsilon' : \mathrm{H}_0(F'(M_j)) \xrightarrow{\cong} R$. Hence, there is a unique $z_j \in \mathrm{H}_0(F'(M_j))$ with $\varepsilon'(z_j) = \varepsilon(d_{0j})$. Pick any $c_{0j} \in Z_j$ and define $\Phi_0(M_j)(d_{0j}) = c_{0j}$. In particular, if $\varepsilon(d_{0j}) = 0$, then $z_j = 0$, and hence $c_{0j} \in \operatorname{im}(\partial_1')$. Hence, $\Phi_0(M_j)(\partial_1 d_{1j}) = \partial_1'(c_{1j})$ for some $c_{1j} \in F_1'(M_j)$. Define $\Phi_1(M_j)(d_{1j}) = c_{1j}$, in particular this gives $\Phi_0(M_j)\partial_1 = \partial_1'\Phi_1(M_j)$.

Assume by induction that we have defined $\Phi_p(M_j)$ for $p < k$. Since $\mathrm{H}_{k-1}(F'(M_j)) = 0$ and $\partial_{k-1}'\Phi_{k-1}(\partial_{kd_{kj}}) = \Phi_{k-2}(\partial_{k-1}\partial_{kd_{kj}}) = 0$, we must have $c_{kj} \in F_{k'}(M_j)$ with $\Phi_{k-1}(\partial_k d_{kj}) = \partial_k'(c_{kj})$. Define $\Phi_k(M_j)(d_{kj}) = c_{kj}$, in particular we have $\Phi_{k-1}(M_j)\partial_k = \partial_k'\Phi_k(M_j)$. So we have defined $\Phi(M_j)$. We must show that $\Phi_{k-1}(X)\partial_k = \partial_k'\Phi_k(X)$ for any X, and if $g : X \to Y$ is any map, then the following commutes

$$\begin{tikzcd} F_k(X) \arrow[r, "\Phi_k(X)"] \arrow[d, "F_k(g)"'] & F_k'(X) \arrow[d, "F_k'(g)"] \\ F_k(Y) \arrow[r, "\Phi_k(Y)"] & F_k'(Y) \end{tikzcd}$$

$$\begin{aligned}
\partial\Phi_k(X)F_k(f)(d_{kj}) &= \partial F_k'(f)\Phi_k(M_j)(d_{kj}) && \text{(by definition of } \Phi_k(X)) \\
&= F_{k-1}'(f)\partial\Phi_k(M_j)(d_{kj}) && \text{(since } F'(g) \text{ chain map)} \\
&= F_{k-1}'(f)\Phi_{k-1}(M_j)\partial(d_{kj}) && \\
&= \Phi_{k-1}(X)F_{k-1}(f)\partial(d_{kj}) && \text{(by definition of } \Phi_{k-1}(X)) \\
&= \Phi_{k-1}(X)\partial F_k(f)(d_{kj}) && (F(f) \text{ is a chain map})
\end{aligned}$$

Hence $\partial\Phi_k(X) = \Phi_{k-1}(X)\partial$.

$$
\begin{aligned}
F'_k(g)\Phi_k(X)F_k(f)(d_{kj}) &= F'_k(g)F'_k(f)\Phi_k(M_j)(d_{kj}) \\
&= F'_k(g\circ f)\Phi_k(M_j)(d_{kj}) \\
&= \Phi_k(Y)F_k(g\circ f)(d_{kj}) \\
&= \Phi_k(Y)F_k(g)F_k(f)(d_{kj})
\end{aligned}
$$

so $F'_k(g)\circ\Phi_k(X) = \Phi_k(Y)\circ F_k(g)$.

Now, assume $\Psi : F \to F'$ is any other morphism. We must show that $\exists$ chain homotopy D between Φ and Ψ, i.e. for every $X \in \mathrm{obj}(\mathcal{A})$ there is $D_k(X) : F_k(X) \to F'_{k+1}(X)$ such that if $f : Y \to X$ is any map the following commutes

$$
\begin{tikzcd}
F_k(Y) \arrow[r, "D_k(Y)"] \arrow[d, "F_k(f)"'] & F'_{k+1}(Y) \arrow[d, "F'_{k+1}(f)"] \\
F_k(X) \arrow[r, "D_k(X)"'] & F'_{k+1}(X)
\end{tikzcd}
$$

and $\partial D_k(X) + D_{k-1}(X)\partial = \Phi_k(X) - \Psi_k(X)$. Again by taking $Y = M_j$ we get

$$D_k(X)F_k(f)(d_{kj}) = F'_{k+1}(f)D_k(M_j)(d_{kj})$$

so it suffices to define $D_k(M_j)(d_{kj})$. Since

$$
\begin{tikzcd}
F_0(M_j) \arrow[rr, two heads, "\varepsilon"] \arrow[d, bend right, "\Psi_0(M_j)"'] \arrow[d, bend left, "\Phi_0(M_j)"] & & R \\
F'_0(M_j) \arrow[urr, two heads, "\varepsilon'"'] & &
\end{tikzcd}
$$

commutes $\varepsilon'(\Phi_0(M_j)(d_{0j}) - \Psi_0(M_j)(d_{0j})) = 0$. Since $\ker(\varepsilon') = \mathrm{im}(\partial'_1)$, $\exists e_{1j} \in F'_1(M_j)$

with $\partial'_1(e_{1j}) = \Phi_0(M_j)(d_{0j}) - \Psi_0(M_j)(d_{0j})$. Define $D_0(M_j)(d_{0j}) = e_{1j}$, so $\partial D_0(M_j) = \Phi_0(M_j) - \Psi_0(M_j)$. By induction assume we have defined $D_p(X)$ for all $p < k$. Since $\mathrm{H}_k(F'(M_j)) = 0$, and

$$\begin{aligned}
&\partial'_k(\Phi_k(M_j)(d_{kj}) - \Psi_k(M_j)(d_{kj}) - D_{k-1}(M_j)(\partial d_{kj})) \\
=&\Phi_{k-1}(M_j)(\partial_k d_{kj}) - \Psi_{k-1}(M_j)(\partial_k d_{kj}) - \partial'_k D_{k-1}(M_j)(\partial d_{kj}) = 0
\end{aligned}$$

since by induction

$$\partial'_k D_{k-1}(M_j)(\partial d_{kj}) = \Phi_{k-1}(M_j)(\partial d_{kj}) - \Psi_{k-1}(M_j)(\partial d_{kj}) - \underbrace{D_{k-2}(M_j)\partial\partial d_{kj}}_{0},$$

$\exists e_{k+1,j} \in F'_{k+1}(M_j)$ so that

$$\partial' e_{k+1,j} = \Phi_k(M_j)(d_{kj}) - \Psi_k(M_j)(d_{kj}) - D_{k-1}(M_j)(\partial d_{kj})$$

Define $D_k(M_j)(d_{kj}) = e_{k+1,j}$, hence we have

$$\partial D_k(M_j) = \Phi_k(M_j) - \Psi_k(M_j) - D_{k-1}(M_j)\partial.$$

Hence

$$\begin{aligned}
\partial D_k(X)F_k(f)(d_{kj}) &= \partial F'_{k+1}(f)D_k(M_j)(d_{kj}) \\
&= F'_k(f)\partial D_k(M_j)(d_{kj}) \\
&= F'_k(f)(\Phi_k(M_j) - \Psi_k(M_j) - D_{k-1}(M_j)\partial)(d_{kj}) \\
&= \Phi_k(X)F_k(f)(d_{kj}) - \Psi_k(X)F_k(f)(d_{kj}) - D_{k-1}(X)F_{k-1}(f)(\partial d_{kj}) \\
&= (\Phi_k(X) - \Psi_k(X) - D_{k-1}(X)\partial)\, F_k(f)(d_{kj}) \\
\therefore\ \partial D_k(X) &= \Phi_k(X) - \Psi_k(X) - D_{k-1}(X)\partial
\end{aligned}$$

Finally, we must show if $g : X \to Y$

$$\begin{aligned} D_k(Y)F_k(g)F_k(f)(d_{kj}) &= D_k(Y)F_k(g \circ f)(d_{kj}) \\ &= F'_{k+1}(g \circ f)D_k(M_j)(d_{kj}) \\ &= F'_{k+1}(g)F'_{k+1}(f)D_k(M_j)(d_{kj}) \\ &= F'_{k+1}(g)D_k(X)F_k(f)(d_{kj}) \\ \therefore\ D_k(Y)F_k(g) &= F'_{k+1}(g)D_k(X) \end{aligned}$$

i.e.

$$\begin{tikzcd} F_k(X) \arrow[r, "D_k(X)"] \arrow[d, "F_k(g)"'] & F'_{k+1}(X) \arrow[d, "F'_{k+1}(g)"] \\ F_k(Y) \arrow[r, "D_k(Y)"'] & F'_{k+1}(Y) \end{tikzcd}$$

commutes. □

2.1.1 Alexander chain map

Theorem 2.2. *Let $\mathcal{P}$ be a category of ordered pairs of topological spaces, and $\mathcal{C}$ be the category of augmented chain complexes. Then the functors $F, F' : \mathcal{P} \to \mathcal{C}$ $\ F(X,Y) = S(X \times Y), F'(X,Y) = S(X) \otimes S(Y)$ are chain homotopic. In particular $\widetilde{H}_*(X \times Y) = \widetilde{H}_*(S(X) \otimes S(Y))$.*

Proof. Pick $\mathcal{M} = \{(\Delta_k, \Delta_l)\}$ to be the models in $\mathcal{P}$. Let $d_k : \Delta_k \to \Delta_k \times \Delta_k$ be the diagonal map. Then any singular simplex $\sigma : \Delta_k \to X \times Y$ can be written $\sigma = (\sigma' \times \sigma'') \circ d_k : \Delta_k \xrightarrow{d_k} \Delta_k \times \Delta_k \xrightarrow{\sigma' \times \sigma''} X \times Y$ where $\sigma = (\sigma', \sigma'')$. We can write $\sigma = S_k(\sigma' \times \sigma'')(d_k)$. So $\{d_k \in S_k(\Delta_k \times \Delta_k)\}$ is a basis for $F_k(X,Y)$. Hence F is free (over $\mathcal{M}$.) Also since $\Delta_k \times \Delta_l$ is contractible $\widetilde{H}_*(F(\Delta_k, \Delta_l)) = 0$, so F' is also acyclic over $\mathcal{M}$. Similarly, since $\{\delta_i \in S_i(\Delta_i)\}$ is a basis for $S_i(X)$ and $\{\delta_j \in S_j(\Delta_l)\}$ is a basis for $S_j(Y)$, $\{\delta_i \otimes \delta_j \in S_i(\Delta_i) \otimes S_j(\Delta_j)\}$ is a basis for $S_i(X) \otimes S_j(Y)$. Hence

$\{\oplus_{i+j=k}\delta_i \otimes \delta_j \in (S(\Delta_i) \otimes S(\Delta_j))_k\}$ is a basis for $(S(X) \otimes S(Y))_k$. So F' is free (over $\mathcal{M}$) and since each Δ_k is contractible $\widetilde{H}_*(F'(\Delta_k, \Delta_l) = 0$. Therefore both F, F' are free and acyclic over $\mathcal{M}$. So there are morphisms $\Phi : F \to F'$, $\Psi : F' \to F$ such that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are chain homotopic to identities. Hence for every (X, Y) we have

$$\widetilde{H}_*(X \times Y) = \widetilde{H}_*(S(X \times Y)) \underset{\Psi_*}{\overset{\Phi_*}{\rightleftarrows}} \widetilde{H}_*(S(X) \otimes S(Y))$$

with $\Phi_* \circ \Psi_* = Id$ and $\Psi_* \circ \Phi_* = Id$. □

Since Φ, Ψ are unique up to chain homotopy, if A is any other morphism $F \to F'$ then $A_* = \Phi_*$. So there exists a natural chain equivalence:

$$S(X \times Y) \to S(X) \otimes S(Y) \tag{2.1}$$

unique up to chain homotopy. Combining this fact with Theorem 1.19 we get:

Corollary 2.3. *For any topological space X there is the following exact sequence*

$$0 \to [H(X) \otimes H(Y)]_n \to H_n(X \times Y) \to [H(X) * H(Y)]_{n-1} \to 0$$

In particular, if all $H(X)$ and $H(Y)$ are free then

$$H_n(X \times Y) \cong [H(X) \otimes H(Y)]_n$$

It is useful to construct a specific chain equivalence realizing (2.1); a useful one is so called Alexander-Whitney chain equivalence discussed in Section 2.1.2.

2.1.2 Alexander-Whitney homomorphism

Definition 2.4 (Alexander-Whitney homomorphism)**.**

$$A_n : S_n(X \times Y) \longrightarrow [S(X) \otimes S(Y)]_n$$

Let $\sigma \in S_n(X \times Y)$ then $\sigma = (\sigma', \sigma'')$ where $\sigma' \in S_n(X)$ and $\sigma'' \in S_n(Y)$. Let

$$A_n(\sigma) = \sum_{k=0}^{n} \sigma' \lambda_k \otimes \sigma'' \rho_{n-k}$$

then extend A_n linearly where

$$\begin{aligned} \lambda_k \quad & : \Delta_k \to \Delta_n \text{ given by } (p_0, \ldots, p_k) \\ \rho_{n-k} \quad & : \Delta_{n-k} \to \Delta_n \text{ given by } (p_k, \ldots, p_n) \end{aligned}$$

Let $A = \{A_n\}$. We claim A is a morphism between $S(X \times Y)$ and $S(X) \otimes S(Y)$, i.e.

1. A is functorial with respect to maps, i.e. if $f : X' \to X, g : Y' \to Y$ are maps then the following diagram commutes

$$\begin{tikzcd} S_n(X' \times Y') \arrow[r, "A_n"] \arrow[d, "S_n(f \times g)"'] & {[S(X') \otimes S(Y')]_n} \arrow[d, "\oplus_{i+j=n} S_i(f) \otimes S_j(g)"] \\ S_n(X \times Y) \arrow[r, "A_n"] & {[S(X) \otimes S(Y)]_n} \end{tikzcd}$$

2. $A_{n+1}\partial_n = \partial_n A_n$

Proof. 1. is trivial.

2. is tedious but we will do it. Take $X' = Y' = \Delta_n$ and $d_n \in S_n(\Delta_n \times \Delta_n)$ be the diagonal, then if $\sigma = (\sigma', \sigma'') \in S_n(X' \times Y')$ then $\sigma = S_n(\sigma' \times \sigma'')(d_n)$. Hence from (1)

$$\begin{aligned} A_n(\sigma) &= A_n S_n(\sigma' \times \sigma'')(d_n) = \left[\bigoplus_{i+j=n} S_i(\sigma') \otimes S_j(\sigma'') \right] A_n(d_n) \\ \partial A_n(\sigma) &= \partial \left[\bigoplus_{i+j=n} S_i(\sigma') \otimes S_j(\sigma'') \right] A_n(d_n) \\ &= \bigoplus_{i+j=n-1} \left[S_i(\sigma') \otimes S_j(\sigma'') \right] \partial A_n(d_n), \end{aligned}$$

$$\begin{aligned}
A_{n-1}(\partial\sigma) &= A_{n-1}\left(\partial S_n(\sigma' \times \sigma'')(d_n)\right)\\
&= A_{n-1}\left(S_{n-1}(\sigma' \times \sigma'')\partial d_n\right)\\
&= \bigoplus_{i+j=n-1} S_i(\sigma' \otimes \sigma'')A_{n-1}(\partial d_n).
\end{aligned}$$

Hence it suffices to prove $\partial A_n(d_n) = A_{n-1}(\partial d_n)$.

$$\begin{aligned}
\partial A_n(d_n) &= \partial A_n(Id, Id) = \partial\left(\sum_{k=0}^{n} \lambda_k \otimes \rho_{n-k}\right)\\
&= \sum_{k=0}^{n}\left(\partial\lambda_k \otimes \rho_{n-k} + (-1)^k \lambda_k \otimes \partial\rho_{n-k}\right)\\
&= \sum_{k=0}^{n}\left(\sum_{i=0}^{k}(-1)^i \lambda_k F_k^i \otimes \rho_{n-k} + \sum_{j=0}^{n-k}(-1)^{k+j}\lambda_k \otimes \rho_{n-k}F_{n-k}^j\right)\\
&= \sum_{k=0}^{n}\left(\sum_{i=0}^{k}(-1)^i F_n^i \lambda_{k-1} \otimes \rho_{n-k} + \sum_{j=0}^{n-k}(-1)^{k+j}\lambda_k \otimes F_n^{j+k}\rho_{n-k-1}\right)
\end{aligned}$$

since the following diagrams commute

$$\begin{tikzcd}
\Delta_{k-1} \arrow[r, "F_k^i"] \arrow[d, "\lambda_{k-1}"'] & \Delta_k \arrow[d, "\lambda_k"] \\
\Delta_{n-1} \arrow[r, "F_n^i"'] & \Delta_n
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta_{n-k} \arrow[d, "\rho_{n-k}"'] & \Delta_{n-k-1} \arrow[l, "F_{n-k}^j"'] \arrow[d, "\rho_{n-k-1}"] \\
\Delta_n & \Delta_{n-1} \arrow[l, "F_n^{j+k}"]
\end{tikzcd}$$

$$\text{where:} \quad F_k^i(v_j) = \begin{cases} v_j & j < i \\ v_{j+1} & j \geq i \end{cases}$$

Continuing the above calculation, we have

$$\begin{aligned}
&= \sum_{k=0}^{n}\left(\sum_{i=0}^{k}(-1)^i F_n^i\lambda_{k-1}\otimes\rho_{n-k} + \sum_{j=k}^{n}(-1)^j\lambda_k\otimes F_n^j\rho_{n-k-1}\right)\\
&= \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^i F_n^i\lambda_{k-1}\otimes\rho_{n-k} + \sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^j\lambda_k\otimes F_n^j\rho_{n-k-1}
\end{aligned}$$

$$\begin{aligned}
A_{n-1}(\partial d_n) &= A_{n-1}(\sum_{k=0}^{n}(-1)^k d_n\circ F_n^k) = A_{n-1}\left(\sum_{k=0}^{n}(-1)^k(F_n^k,F_n^k)\right)\\
&= \sum_{k=0}^{n}(-1)^k\left(\sum_{r=0}^{n-1}F_n^k\circ\lambda_r\otimes F_n^k\circ\rho_{n-1-r}\right)\\
&= \sum_{r=1}^{n}\left(\sum_{k=0}^{n}(-1)^k F_n^k\lambda_{r-1}\otimes F_n^k\rho_{n-r}\right)\\
&= \sum_{r=1}^{n}\sum_{k=0}^{r-1}(-1)^k F_n^k\lambda_{r-1}\otimes\rho_{n-r} + \sum_{r=1}^{n}\sum_{k=r}^{n}(-1)^k\lambda_{r-1}\otimes F_n^k\rho_{n-r}
\end{aligned}$$

Since $F_n^k\lambda_p = \lambda_p$ for $k > p$, $F_n^k\rho_p = \rho_p$ for $k < n-p$,

$$\begin{aligned}
&= \sum_{k=0}^{n-1}\sum_{i=0}^{k}(-1)^i F_n^i\lambda_{k-1}\otimes\rho_{n-k} + \sum_{k=0}^{n-1}\sum_{j=k}^{n}(-1)^j\lambda_k\otimes F_n^j\rho_{n-k-1}\\
&= \sum_{r=0}^{n-1}\sum_{k=0}^{n}(-1)^k F_n^k\lambda_r\otimes F_n^k\rho_{n-1-r}\\
&= \sum_{r=0}^{n-1}\sum_{k=0}^{r}(-1)^k F_n^k\lambda_r\otimes\rho_{n-1-r} + \sum_{r=0}^{n-1}\sum_{k=r+1}^{n}(-1)^k\lambda_r\otimes F_n^k\rho_{n-1-r}\\
&= \sum_{k=0}^{n-1}\sum_{i=0}^{k}(-1)^i F_n^i\lambda_k\otimes\rho_{n-1-k} + \sum_{k=0}^{n-1}\sum_{j=k+1}^{n}(-1)^j\lambda_k\otimes F_n^j\rho_{n-1-k}\\
&= \sum_{k=1}^{n}\sum_{i=0}^{k-1}(-1)^i F_n^i\lambda_{k-1}\otimes\rho_{n-k} + \sum_{k=0}^{n-1}\sum_{j=k+1}^{n}(-1)^j\lambda_k\otimes F_n^j\rho_{n-1-k} \quad \square
\end{aligned}$$

2.2 Cross product

Cross product operation in cohomology is induced by the Alexander chain map as follows

$$\begin{array}{rcl} \mathrm{H}^n(X) \times \mathrm{H}^m(Y) & \longrightarrow & \mathrm{H}^{n+m}(X \times Y) \\ (\alpha, \beta) & \longmapsto & \alpha \times \beta \end{array}$$

$$\begin{tikzcd} S(X \times Y) \arrow[r, "A"] \arrow[rrr, bend right, "\alpha \times \beta = A^*(\alpha \otimes \beta)"'] & S(X) \otimes S(Y) \arrow[r, "\alpha \otimes \beta"] & R \otimes R \arrow[r, "m"] & R \end{tikzcd}$$

Proposition 2.5. $\delta(\alpha \times \beta) = \delta\alpha \times \beta + (-1)^p \alpha \times \delta\beta$

Proof.

$$\begin{array}{rcl} \langle \delta(\alpha \times \beta), a \rangle & = & \langle A^*(\alpha \otimes \beta), \partial a \rangle \\ & = & \langle \alpha \otimes \beta, A(\partial a) \rangle \\ & = & \langle \alpha \otimes \beta, \partial A(a) \rangle \\ & = & \langle \delta(\alpha \otimes \beta), A(a) \rangle \\ & = & \langle \delta\alpha \otimes \beta + (-1)^p \alpha \otimes \delta\beta, A(a) \rangle \\ & = & \langle A^*(\delta\alpha \otimes \beta) + (-1)^p A^*(\alpha \otimes \delta\beta), a \rangle \\ & = & \langle \delta\alpha \times \beta + (-1)^p \alpha \times \delta\beta, a \rangle \quad \square \end{array}$$

Hence the cross product on cochains induce cross product in cohomology classes.

Definition 2.6.

$$\begin{array}{rcl} C : S_p(X) \otimes S_q(Y) & \longrightarrow & S_q(Y) \otimes S_p(X) \\ \alpha \otimes \beta & \longmapsto & (-1)^{pq} \beta \otimes \alpha \end{array}$$

C is a chain map since:

$$\begin{aligned}
C\partial(\alpha\otimes\beta) &= C\left(\partial\alpha\otimes\beta+(-1)^p\alpha\otimes\partial\beta\right)\\
&= (-1)^{(p-1)q}\beta\otimes\partial\alpha+(-1)^{p+p(q-1)}\partial\beta\otimes\alpha\\
&= (-1)^{pq}\left[\partial\beta\otimes\alpha+(-1)^q\beta\otimes\partial\alpha\right]\\
&= (-1)^{pq}\partial(\beta\otimes\alpha)\\
&= \partial C(\alpha\otimes\beta).
\end{aligned}$$

Let $\varphi : X\times Y\to Y\times X$ be the map $x\times y\mapsto y\times x$, then

$$\begin{tikzcd}
 & S(X)\otimes S(Y) \arrow[rr, "C"] & & S(Y)\otimes S(X) \arrow[d, "\alpha\otimes\beta"] \\
S(X\times Y) \arrow[ur, "A"] \arrow[dr, "S(\varphi)"'] & & & R\otimes_R R\cong R \\
 & S(Y\times X) \arrow[uurr, "A"'] & &
\end{tikzcd}$$

Since any two chain maps $S(X\times Y)\to S(X)\otimes S(Y)$ chain homotopic we conclude

$$\varphi^*(\alpha\times\beta)=(-1)^{pq}(\beta\times\alpha)$$

which comes from $\varphi^*A^*(\alpha\otimes\beta)=(-1)^{pq}A^*(\beta\otimes\alpha)$

2.3 Cup product

$$\mathrm{H}^n(X)\times\mathrm{H}^m(X)\to\mathrm{H}^{n+m}(X)$$

$$(\alpha,\beta)\longmapsto\alpha\smile\beta$$

$$S(X)\xrightarrow[d_*]{S(d)}S(X\times X)\xrightarrow{A}S(X)\otimes S(X)\xrightarrow{\alpha\otimes\beta}R\otimes R\xrightarrow{m}R$$

Cup product is the composition $\alpha \smile \beta = d^*(\alpha \times \beta) = d^* A^*(\alpha \otimes \beta) = (\alpha \otimes \beta) \circ A \circ S(d)$

$$\begin{aligned}
\langle \alpha \smile \beta, \sigma \rangle &= \langle d^*(\alpha \times \beta), \sigma \rangle = \langle \alpha \times \beta, d_*(\sigma) \rangle \\
&= \langle \alpha \times \beta, (\sigma, \sigma) \rangle = \langle A^*(\alpha \otimes \beta), (\sigma, \sigma) \rangle \\
&= \langle \alpha \otimes \beta, A_*(\sigma, \sigma) \rangle \\
&= \left\langle \alpha \otimes \beta, \Sigma_{k=0}^{p+q} \lambda_k \otimes \rho_{p+q-k} \right\rangle \\
&= \langle \alpha, \lambda_q \rangle \langle \beta, \rho_q \rangle
\end{aligned}$$

Proposition 2.7. *The cup product satisfies the following properties*

1. $\alpha \smile \beta = (-1)^{pq}(\beta \smile \alpha)$
2. *If* $f : X \to Y$ *is a map then* $f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta$
3. $\alpha \times \beta = p_1^*(\alpha) \smile p_2^*(\beta)$ *where* $p_1 : X \times Y \to X$ *and* $p_2 : X \times Y \to Y$ *are the obvious projection maps.*
4. *Let* $\alpha \in H^n(X), \alpha' \in H^m(X)$ *and* $\beta \in H^p(Y), \beta' \in H^q(Y)$*, then*

$$(\alpha \smile \alpha') \times (\beta \smile \beta') = (-1)^{mp}(\alpha \times \beta) \smile (\alpha' \times \beta')$$

5. $1 \times \alpha = \alpha \times 1 = \alpha$ *and* $1 \smile \alpha = \alpha \smile 1 = \alpha$

Proof. All the proofs follow from definitions and application of Theorem 2.2:

1.

$$\begin{aligned}
\alpha \smile \beta &= d^*(\alpha \times \beta) \\
&= d^*\left((-1)^{pq} \beta \times \alpha\right) \\
&= (-1)^{pq} d^*(\beta \times \alpha) \\
&= (-1)^{pq} \beta \smile \alpha
\end{aligned}$$

2.

$$\begin{aligned}\langle f^*(\alpha \smile \beta), a\rangle &= \langle \alpha \smile \beta,\ f_* a\rangle \\ &= \langle \alpha, f\circ a\circ \lambda_p\rangle\,\langle \beta, f\circ a\circ \rho_q\rangle \\ &= \langle f^*\alpha, a\circ \lambda_p\rangle\,\langle f^*\beta, a\circ \rho_q\rangle \\ &= \langle f^*\alpha \smile f^*\beta, a\rangle\end{aligned}$$

3.

$$\begin{tikzcd} S(X\times Y) \arrow[r, "A"] \arrow[r, bend right, "S(p_1)\otimes S(p_2)"'] & S(X)\otimes S(Y) \arrow[r, "\alpha\otimes\beta"] & R\otimes R \arrow[r] & R \end{tikzcd}$$

$$\implies \alpha\times\beta = p_1^*(\alpha) \smile p_2^*(\beta)$$

4.

$$\begin{tikzcd} & & S(X\times X)\otimes S(Y\times Y) \arrow[d, "A\otimes A"] & R \\ & S(X)\otimes S(Y) \arrow[r] \arrow[ur, bend left, "S(d)\otimes S(d)"] \arrow[ddr, bend right] & S(X)\otimes S(X)\otimes S(Y)\otimes S(Y) \arrow[ur, "\alpha\otimes\alpha'\otimes\beta\otimes\beta'"'] & \\ S(X\times Y) \arrow[ur, "A"] \arrow[d, "S(d)"'] & & S(X)\otimes S(Y)\otimes S(X)\otimes S(Y) \arrow[u, "1\otimes C\otimes 1"'] & \\ S((X\times Y)\times(X\times Y)) \arrow[rr, "A"] & & S(X\times Y)\otimes S(X\times Y) \arrow[u, "A\otimes A"'] & \end{tikzcd}$$

By Theorem 2.2 the above diagram of maps commuting up to chain homotopy so

$$\begin{aligned}(1\otimes C\otimes 1)\circ(A\otimes A)\circ A\circ S(d) &\simeq (A\otimes A)\circ(S(d)\otimes S(d))\circ A \\ &\simeq (A\circ S(d))\otimes(A\circ S(d))\circ A\end{aligned}$$

$$
\begin{aligned}
\implies \quad & A^*\left(d^*A^*(\alpha\otimes\alpha')\otimes d^*A^*(\beta\otimes\beta')\right) = \\
= \ & (\alpha\smile\alpha')\times(\beta\smile\beta') \\
= \ & (-1)^{mp}(\alpha\times\beta)\smile(\alpha'\times\beta') \\
= \ & (-1)^{mp}d^*A^*\left(A^*(\alpha\otimes\beta)\otimes A^*(\alpha'\otimes\beta')\right) \\
= \ & (-1)^{mp}d^*A^*\left(A^*\otimes A^*\right)\left(\alpha\otimes\beta\otimes\alpha'\otimes\beta'\right)) \\
= \ & (-1)^{mp}d^*A^*\left(A^*\otimes A^*\right)\left(\alpha\otimes\beta\otimes\alpha'\otimes\beta'\right)) \\
= \ & (-1)^{mp}d^*A^*\left(A^*\otimes A^*\right)\left(1\otimes C\otimes 1\right)\left(\alpha\otimes\alpha'\otimes\beta\otimes\beta'\right))
\end{aligned}
$$

Another proof of (4) is as follows:

$$
\begin{aligned}
(a\smile c)\times(b\smile d) &= p_1^*(a\smile c)\smile p_2^*(b\smile d) \\
&= p_1^*(a)\smile p_1^*(c)\smile p_2^*(b)\smile p_2^*(d) \\
&= (-1)^{|b||c|}p_1^*(a)\smile p_2^*(b)\smile p_1^*(c)\smile p_2^*(d) \\
&= (-1)^{|b||c|}(a\times b)\smile(c\times d) \quad \square
\end{aligned}
$$

Example 2.2. *Let $X = S^p\vee S^q\vee S^{p+q}$, $Y = S^p\times S^q$, then $H^*(X) = H^*(Y)$, but their cup product structures are different. Consider the following inclusions and projections*

$$
\begin{tikzcd}
S^p \arrow[r, hook, "i"] & X & S^q \arrow[l, hook', "j"'] \\
& S^{p+q} \arrow[u, hook, "k"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S^p\times S^q \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
S^p & & S^q
\end{tikzcd}
$$

For $k = p, q$ let α_k, β_k be generators of $H^k(X)$, $H^k(Y)$ respectively, then

- $k^* : H^{p+q}(X) \xrightarrow{\approx} H^{p+q}(S^{p+q})$
- $k^*(\alpha_p\smile\alpha_q) = k^*(\alpha_p)\smile k^*(\alpha_q) = 0 \implies \alpha_p\smile\alpha_q = 0$
- $\beta_p\smile\beta_q = p_1^*([S^p])\smile p_2^*([S^q]) = \beta_p\times\beta_q \neq 0$

2.4 Cap Product

$$H^p(X) \otimes H_n(X) \xrightarrow{\frown} H_{n-p}(X) \quad \text{, where } n \geq p$$

Cap product is first defined on chain level then checked that it is defined on homology

$$S^p(X) \otimes S_n(X) \xrightarrow{\frown} S_{n-p}(X)$$

$$\left\langle b^{n-p}, c^p \frown \sigma_n \right\rangle := \left\langle b^{n-p} \smile c^p, \sigma_n \right\rangle$$

From this we get $\partial(c^p \frown \sigma_{p+q}) = (-1)^q \delta c \frown \sigma + c \frown \partial\sigma$ since :

$$\begin{aligned}
\left\langle \alpha^{q-1}, \partial(c^p \frown \sigma_{p+q}) \right\rangle &= \left\langle \delta\alpha^{q-1}, c^p \frown \sigma_{p+q} \right\rangle \\
&= \left\langle \delta\alpha^{q-1} \smile c^p, \sigma_{p+q} \right\rangle \\
&= \left\langle \delta(\alpha^{q-1} \smile c^p) + (-1)^q \alpha \smile \delta c, \sigma_{p+q} \right\rangle \\
&= \left\langle \alpha^{q-1} \smile c^p, \partial\sigma \right\rangle + (-1)^q \left\langle \alpha, \delta c \frown \sigma \right\rangle \\
&= \left\langle \alpha^{q-1}, c^p \frown \partial\sigma \right\rangle + (-1)^q \left\langle \alpha, \delta c \frown \sigma \right\rangle \\
&= \left\langle \alpha, c^p \frown \partial\sigma + (-1)^q \delta c \frown \sigma_{p+q} \right\rangle
\end{aligned}$$

The above definition is equivalent to

$$c^p \frown \sigma_{p+q} = \langle c^p, \sigma \circ \rho_p \rangle \, \sigma \circ \lambda_q \quad \text{, since}$$

$$\begin{aligned}
\langle b^q, c^p \frown \sigma \rangle &= \langle b^q \smile c^p, \sigma \rangle \\
&= \langle b^q, \sigma \circ \lambda_q \rangle \langle c^p, \sigma \circ \rho_p \rangle \\
&= \langle b^q, \langle c^p, \sigma \circ \rho_p \rangle \, \sigma \circ \lambda_q \rangle
\end{aligned}$$

Proposition 2.8. *If $f : X \to Y$, then*

$$f_* \left(f^* \beta^p \frown \sigma_{p+q} \right) = \beta^p \frown f_*(\sigma_{p+q})$$

Proof.

$$\begin{aligned}\langle \alpha, f_*(f^*\beta \frown \sigma)\rangle &= \langle f^*\alpha, f^*\beta \frown \sigma\rangle \\ &= \langle f^*\alpha \smile f^*\beta, \sigma\rangle \\ &= \langle f^*(\alpha \smile \beta), \sigma\rangle \\ &= \langle \alpha \smile \beta, f_*(\sigma)\rangle \\ &= \langle \alpha, \beta \frown f_*(\sigma)\rangle \quad \square\end{aligned}$$

Proposition 2.9. $(\alpha \smile \beta) \frown z = \alpha \frown (\beta \frown z)$

Proof.

$$\begin{aligned}\langle \gamma, (\alpha \smile \beta) \frown z\rangle &= \langle \gamma \smile (\alpha \smile \beta), z\rangle \\ &= \langle \gamma \smile \alpha, \beta \frown z\rangle \\ &= \langle \gamma, \alpha \frown (\beta \frown z)\rangle \quad \square\end{aligned}$$

Proposition 2.10. *(a)* $H^p(X,A) \otimes H_{p+q}(X, A \cup B) \xrightarrow{\frown} H_q(X,B)$

(b) $H^p(X,A) \otimes H^q(X,B)) \xrightarrow{\smile} H^{p+q}(X, A \cup B)$

(c) $H^p(X,A) \otimes H^q(Y,B)) \xrightarrow{\times} H^{p+q}((X,A) \times (Y,B))$
where, $(X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y)$.

Proof. (c) Since $\{X \times B, A \times Y\}$ is excisive couple $\exists$ chain equivalence (excision):

$$\begin{array}{ccc} S(X \times Y)/\left[S(X \times B) + S(A \times Y)\right] & \longrightarrow & S(X \times Y)/S\left(X \times B \cup A \times Y\right) \\ \big\downarrow A & & \\ S(X) \otimes S(Y)/[S(X) \otimes S(B) + S(A) \otimes S(Y)] & \approx & S(X)/S(A) \otimes S(Y)/S(B) \end{array}$$

Therefore we have $S\left((X,A) \times (Y,B)\right) \xrightarrow{A} S(X,A) \otimes S(Y,B) \xrightarrow{\alpha \otimes \beta} R \otimes R$

(b)
$$
\begin{tikzcd}
S(X)/S(A \cup B) \arrow[r, "S(d)"] \arrow[dr, dashed] & S((X,A) \times (X,B)) \arrow[d, "A"] \\
 & S(X,A) \otimes S(X,B)
\end{tikzcd}
$$

(a) For $\alpha^p \otimes \sigma_{p+q} \in Z^p(X,A) \otimes Z_{p+q}(X, A \cup B)$,

$$\partial(\alpha \frown \sigma) = (-1)^q \left[\delta\alpha \frown \sigma + (-1)^q \alpha \frown \partial\sigma\right]$$

where $\partial\sigma \in S_*(A \cup B)$. Note that $\alpha \frown \partial\sigma = \langle \alpha^p, \partial\sigma \circ \rho_p \rangle\, \partial\sigma \circ \lambda_q \in S_*(B)$. Also, note that $\delta\alpha \frown \sigma \in S_*(B)$ since

$$
\begin{aligned}
\langle \beta, \delta\alpha \frown \sigma \rangle &= \langle \beta \smile \delta\alpha, \sigma \rangle = \pm \langle \delta\alpha, \beta \frown \sigma \rangle \\
&= \pm \langle \alpha, \partial(\beta \frown \sigma) \rangle \\
&= \pm \langle \alpha, \langle \beta, \sigma \circ \rho \rangle\, \partial(\sigma \circ \lambda) \rangle \in S_*(B)
\end{aligned}
$$

since $\partial(\sigma \circ \lambda) \in S_*(A)$. Therefore, $\partial(\alpha \frown \sigma) \in S_*(B)$. $\square$

Proposition 2.11. *If $\sigma \in H_{p+q}(X, A)$ then*

$$
\begin{tikzcd}
H^p(X,A) \arrow[r, "j^*"] \arrow[d, "\frown \sigma"] & H^p(X) \arrow[d, "\frown \sigma"] \\
H_q(X) \arrow[r, "j_*"] & H_q(X,A)
\end{tikzcd}
$$

where $A \underset{i}{\subsetneq} X \underset{j}{\subsetneq} (X, A)$ and $\sigma = \sigma_{p+q}$

If $A \subset B \subset X$, then

$$
\begin{tikzcd}
H^p(X,B) \arrow[r, "i^*"] \arrow[d, "\frown \sigma"] & H^p(X,A) \arrow[d, "\frown \sigma"] \\
H_q(X) \arrow[r, equal] & H_q(X)
\end{tikzcd}
$$

where $i : (X,A) \hookrightarrow (X,B)$

Lemma 2.12. *For $\alpha \in H^p(X), a \in H_q(X), \beta \in H^r(Y)$ and $b \in H_s(Y)$,*

$$(\alpha \times \beta) \frown (a \times b) = (-1)^{p(s-r)}(\alpha \frown a) \times (\beta \frown b)$$

Proof. Write $\gamma = \gamma_1 \times \gamma_2, \gamma \in \mathrm{H}^k(X \times Y), \deg(\gamma_i) = k_i$.

$$\begin{aligned}
\langle \gamma_1 \times \gamma_2, (\alpha \times \beta) \frown (a \times b)\rangle &= \langle (\gamma_1 \times \gamma_2) \smile (\alpha \times \beta), a \times b\rangle \\
&= (-1)^{pk_2} \langle (\gamma_1 \smile \alpha) \times (\gamma_2 \smile \beta), a \times b\rangle \\
&= (-1)^{pk_2} \langle \gamma_1 \smile \alpha, a\rangle \langle \gamma_2 \smile \beta, b\rangle
\end{aligned}$$

$$\begin{aligned}
\langle \gamma_1 \times \gamma_2, (\alpha \frown a) \times (\beta \frown b)\rangle &= \langle \gamma_1, \alpha \frown a\rangle \langle \gamma_2, \beta \frown b\rangle \\
&= \left\langle \gamma_1^{k_1} \smile \alpha^p, a\right\rangle \left\langle \gamma_2^{k_2} \smile \beta^r, b\right\rangle
\end{aligned}$$

where $k_1 + p = q$ and $k_2 + r = s$. □

2.5 Slant Product

$$S^n(X \times Y) \times S_p(Y) \xrightarrow{/} S^{n-p}(X)$$

$$c^* \otimes c \longmapsto c^*/c$$

$$\text{where} \quad \langle c^*/c, a\rangle := \langle A(c^*), a \otimes c\rangle \quad \forall a \in S_{n-p}(X)$$

It is easy to check that $\delta(c^*/c) = \delta c^*/c - (-1)^{n-p} c^*/\delta c$, hence we get a map

$$\begin{array}{ccccc}
H^n(X \times Y) & \otimes & H_p(Y) & \longrightarrow & H^{n-p}(X) \\
\alpha & \otimes & c & \longmapsto & c^*/c
\end{array}$$

Relative Version :

$$\mathrm{H}^n\left((X, A) \times (Y, B)\right) \otimes \mathrm{H}_p(Y, B) \longrightarrow \mathrm{H}^{n-p}(X, A)$$

Recall the Alexander chain equivalence

$$S(X)/S(A) \otimes S(Y)/S(B) \longrightarrow S(X \times Y)/S\,(X \times B \cup A \times Y)$$

So, slant product induces a map

$$\mathrm{Hom}\,(S(X)/S(A) \otimes S(Y)/S(B)) \otimes (S(Y)/S(B)) \to \mathrm{Hom}\,(S(X)/S(A))$$

Claim: For $c_1^* \otimes c_2^* \in \mathrm{Hom}\,(S(X)/S(A) \otimes S(Y)/S(B))$ and $c \in S(Y)/S(B)$,

(a) for $d \in S(B)$,

$$c_1^* \otimes c_2^*/c = c_1^* \otimes c_2^*/c + d$$

(b) $c_1^* \otimes c_2^*/c = 0$ on $S(A)$, that is $c_1^* \otimes c_2^*/c \in \mathrm{Hom}(S(X)/S(A))$.

Proof. (a)

$$\begin{aligned}
\langle c_1^* \otimes c_2^*/c + d, a\rangle &= \langle c_1^* \otimes c_2^*, a \otimes (c+d)\rangle \\
&= \langle c_1^* \otimes c_2^*, a \otimes c\rangle + \langle c_1^* \otimes c_2^*, a \otimes d\rangle \\
&= \langle c_1^* \otimes c_2^*/c, a\rangle + \langle c_1^*, a\rangle\,\langle c_2^*, d\rangle \\
&= \langle c_1^* \otimes c_2^*/c, a\rangle \quad \text{since} \quad \langle c_2^*, d\rangle = 0.
\end{aligned}$$

(b) If $a \in S(A)$, then

$$\begin{aligned}
\langle c_1^* \otimes c_2^*/c, a\rangle &= \langle c_1^* \otimes c_2^*, a \otimes c\rangle \\
&= \langle c_1^*, a\rangle\,\langle c_2^*, c\rangle = 0 \;\; \text{since} \;\; \langle c_1^*, a\rangle = 0 \quad \square
\end{aligned}$$

Proposition 2.13. *1. For $\alpha \in H^r(X), \beta \in H^p(Y)$ and $b \in H_p(Y)$;*

$$(\alpha \times \beta)/b = \alpha\,\langle \beta, b\rangle\,.$$

2. *For* $\eta \in H^p(X), \alpha \in H^q(X \times Y), \beta \in H^r(Y)$ *and* $b \in H_s(Y)$*;*

$$\eta \smile [\alpha/\beta \frown b] = (-1)^{pq} [\alpha \smile (\eta \times \beta)] /b.$$

3. *If* $f : X \to X', g : Y \to Y', \alpha \in H^*(X' \times Y')$*, and* $b \in H_*(Y)$ *then*

$$(f \times g)^* \alpha/b = f^*(\alpha/g_* b).$$

Proof. 1.

$$\begin{aligned} \langle (\alpha \times \beta)/b, a \rangle &= \langle \alpha \times \beta, a \times b \rangle \\ &= \langle \alpha, a \rangle \langle \beta, b \rangle = \langle \alpha \langle \beta, b \rangle, a \rangle \end{aligned}$$

2.

$$\begin{aligned} \langle \eta^p \smile [\alpha^q/\beta^r \frown b_s], a_{p+q-s+r} \rangle &= (-1)^{p(q-s+r)} \langle \alpha/\beta \frown b, \eta \frown a \rangle \\ &= (-1)^{p(q-s+r)} \langle \alpha, (\eta \frown a) \times (\beta \frown b) \rangle \\ &= (-1)^{p(q-s+r)+p(s-r)} \langle \alpha, (\eta \times \beta) \frown (a \times b) \rangle \\ &= (-1)^{pq} \langle \alpha \smile (\eta \times \beta), a \times b \rangle \\ &= (-1)^{pq} \langle [\alpha \smile (\eta \times \beta)]/b, a \rangle \end{aligned}$$

3.

$$\begin{aligned} \langle (f \times g)^* \alpha/b, a \rangle &= \langle (f \times g)^* \alpha, a \times b \rangle \\ &= \langle \alpha, (f \times g)_* (a \times b) \rangle \\ &= \langle \alpha, f_* a \times g_* b \rangle \\ &= \langle \alpha/g_* b \ , f_* a \rangle \\ &= \langle f^*(\alpha/g_* b) \ , a \rangle \quad \square \end{aligned}$$

2.6 Steenrod Squares

Recall the chain map $C : S_*(X) \otimes S_*(X) \to S_*(X) \otimes S_*(X)$ given by $C(\alpha_p \otimes \beta_q) = (-1)^{pq}\beta_q \otimes \alpha_p$. Then in the following diagram we will alter A_0 to CA_0 then consider the resulting chain homotopy

$$\begin{tikzcd} & S_*(X \times X) \arrow[d, "A"', "\cong"] \\ S_*(X) \arrow[ur, "d"] \arrow[r, "A_0"] & S_*(X) \otimes S_*(X) \arrow[loop right, "C"] \end{tikzcd}$$

$CA_0, A_0 : S_*(X) \to S_*(X) \otimes S_*(X)$ are two chain maps, (2.1) implies that there is a degree 1 map $A_1 : S_*(X) \to S_*(X) \otimes S_*(X)$ with

$$CA_0 - A_0 = (C-1)A_0 = \partial A_1 - A_1\partial.$$

Hence, $(C+1)A_1$ is a chain map. Indeed, we can easily check

$$\begin{aligned} \partial\left((C+1)A_1\right) &= (C+1)\partial A_1 \\ &= (C+1)\left((C-1)A_0 + A_1\partial\right) \\ &= (C+1)A_1\partial. \end{aligned}$$

Now, $(C+1)A_1, 0 : S_*(X) \to S_*(X) \otimes S_*(X)$ are two chain maps which implies there is a degree 2 map $A_2 : S_*(X) \to S_*(X) \otimes S_*(X)$ with

$$(C+1)A_1 = \partial A_2 - A_2\partial.$$

Hence, $(C-1)A_2$ is a degree 2 chain map. Indeed, we can easily check

$$\begin{aligned} \partial\left((C-1)A_2\right) &= (C-1)\partial A_2 \\ &= (C-1)\left((C+1)A_1 + A_2\partial\right) \\ &= (C-1)A_2\partial. \end{aligned}$$

Then $(C-1)A_2, 0 : S_*(X) \to S_*(X) \otimes S_*(X)$ are two chain maps. So there is a degree 3 map $A_3 : S_*(X) \to S_*(X) \otimes S_*(X)$ with

$$(C-1)A_2 = \partial A_3 - A_3 \partial.$$

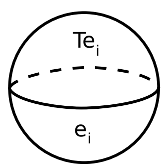
Thus, $(C+1)A_3$ is a degree 3 chain map.

Continuing this way, we get a degree n map for all $n > 0$ with

$$(C + (-1)^n)A_{n-1} = \partial A_n - A_n \partial.$$

We also form the following chain complex (where T is induced by the reflection)

$$W = S_*(S^\infty) = \mathbb{Z}_2 \langle e_i, Te_i \mid i = 0, 1, \dots \rangle.$$



$$\partial e_i = e_{i-1} + Te_{i-1}$$

Proposition 2.14. $\{A_i\}$ *defines* $\mathbb{Z}_2$*-equivariant chain maps (as* $\mathbb{Z}_2$*-chains)*

$$W \otimes S_*(X) \xrightarrow{\varphi} S_*(X) \otimes S_*(X)$$
$$\left\{ \begin{array}{c} \varphi(e_i \otimes \sigma) = A_i(\sigma) \\ \varphi(Te_i \otimes \sigma) = CA_i(\sigma) \end{array} \right\}$$

Proof. Just check φ is a chain map (and recall that we are working mod 2)

$$\begin{aligned} \varphi\partial(e_i \otimes \sigma) &= \varphi(\partial e_i \otimes \sigma + e_i \otimes \partial\sigma) \\ &= \varphi(e_{i-1} \otimes \sigma + Te_{i-1} \otimes \sigma + e_i \otimes \partial\sigma) \\ &= A_{i-1}(\sigma) + CA_{i-1}(\sigma) + A_i(\partial\sigma) \\ &= A_{i-1}(\sigma) + CA_{i-1}(\sigma) + (\partial A_i(\sigma) + (C+1)A_{i-1}(\sigma)) \\ &= \partial A_i(\sigma) \\ &= \partial\varphi(e_i \otimes \sigma) \quad \square \end{aligned}$$

Definition 2.15. Call $\varphi_{e_i}(\sigma) = \varphi(e_i \otimes \sigma)$, and then define

$$S^p(X) \otimes S^q(X) \to S^{p+q-i}(X)$$
$$u \otimes v \mapsto u \smile_i v$$
$$u \smile_i v = A_i^*(u \otimes v) = \varphi_{e_i}^*(u \otimes v)$$

$$\begin{tikzcd} S_*(X) \arrow[r, "A_i"] \arrow[rr, bend right, "u \smile_i v"'] & S_*(X) \otimes S_*(X) \arrow[r, "u_p \otimes v_q"] & R \otimes_R R \cong R \end{tikzcd}$$

Lemma 2.16.

$$\delta(u_p \smile_i v_q) = \delta u \smile_i v + (-1)^p u \smile_i \delta v + (-1)^{i+1} u \smile_{i-1} v + (-1)^{pq+1} v \smile_{i-1} u$$

Proof.

$$\begin{aligned} \langle \delta(u \smile_i v), c \rangle &= \langle u \smile_i v, \partial c \rangle \\ &= (u \otimes v) A_i(\partial c) \\ &= (u \otimes v) \left[\partial A_i - (C + (-1)^i) A_{i-1} \right](c) \\ &= \delta(u \otimes v) A_i(c) - (u \otimes v) C A_{i-1}(c) - (-1)^i (u \otimes v) A_{i-1}(c) \\ &= \left\langle \delta u \smile_i v + (-1)^p u \smile_i \delta v - (-1)^i u \smile_{i-1} v - (-1)^{pq} v \smile_{i-1} u, c \right\rangle \end{aligned}$$

□

Corollary 2.17. *We have the following mod 2 congruences*

$$\delta(u \smile_i v) \equiv \delta u \smile_i v + u \smile_i \delta v + u \smile_{i-1} v + v \smile_{i-1} u$$

$$\delta(u \smile_i u) \equiv \delta u \smile_i u + u \smile_i \delta u$$

Definition 2.18. When $\delta(u) = 0$ we have $\delta(u \smile_i u) \equiv 0$. so we get a map in cohomology

$$\begin{aligned} Sq_i : S^p(X;\mathbb{Z}_2) \to S^{2p-i}(X;\mathbb{Z}_2) \\ Sq_i(u) = u \smile_i u \end{aligned}$$

Proposition 2.19. *$Sq_i : H^p(X;\mathbb{Z}_2) \to H^{2p-i}(X;\mathbb{Z}_2)$ is a homomorphism.*

Proof.

$$\begin{aligned} Sq_i(u+v) &= (u+v) \smile_i (u+v) \\ &= A^*(u \otimes u + v \otimes v + u \otimes v + v \otimes u) \\ &= Sq_i(u) + Sq_i(v) + u \smile_i v + v \smile_i u \\ &= Sq_i(u) + Sq_i(v) \end{aligned}$$

The last equality follows from lemma (2.17). □

Proposition 2.20. *If $f : X \to Y$ continuous, then the diagram below commutes.*

$$\begin{tikzcd} H^p(X;\mathbb{Z}_2) \arrow[r, "Sq_i"] & H^{2p-i}(X;\mathbb{Z}_2) \\ H^p(Y;\mathbb{Z}_2) \arrow[u, "f^*"] \arrow[r, "Sq_i"] & H^{2p-i}(Y;\mathbb{Z}_2) \arrow[u, "f^*"'] \end{tikzcd}$$

Proof. By using naturality of A_i we calculate

$$\begin{aligned} f^*Sq_i(u)(c) &= f^*A_i^*(u \otimes u)(c) \\ &= A_i^*(u \otimes u)f_*(c) \\ &= (u \otimes u)A_i(f_*(c)) \\ &= (u \otimes u)(f^* \otimes f^*)A_i(c) \\ &= (f^*u \otimes f^*u)A_i(c) \\ Sq_i(f^*u)(c) &= A_i^*(f^*u \otimes f^*u)(c) \quad \square \end{aligned}$$

Definition 2.21. Define $Sq^i : \mathrm{H}^p(X;\mathbb{Z}_2) \to \mathrm{H}^{p+i}(X;\mathbb{Z}_2)$ by

$$Sq^i(u) = Sq_{p-i}(u_p) \text{ for } i = 0, \ldots$$

From the definition it follows that:

1. $Sq^i(u_p) = 0$ if $i > p$
2. $Sq^p(u_p) = Sq_0(u_p) = A^*(u_p \otimes u_p) = u_p^2$.

2.6.1 Relative Case:

The inclusions $A \overset{j}{\hookrightarrow} X \overset{q}{\hookrightarrow} (X, A)$ induce short exact sequence of cochain complexes

$$0 \to S^*(X,A) \xrightarrow{q^*} S^*(X) \xrightarrow{j^*} S^*(A) \to 0$$

Lemma 2.22. $j^*(u \smile_i v) = j^*(u) \smile_i j^*(v)$

Proof. We can check this by using naturality of Alexander chain map

$$\begin{tikzcd}
S_*(X) \arrow[r, "A_X"] & S_*(X) \otimes S_*(X) \\
S_*(A) \arrow[u, "j_*"] \arrow[r, "A_Y"] & S_*(A) \otimes S_*(A) \arrow[u, "j_*"]
\end{tikzcd}$$

$$\begin{aligned}
\langle j^*(u \smile_i v), c\rangle &= \langle u \smile_i v, j_* c\rangle \\
&= \langle u \otimes v, A_X(j_* c)\rangle \\
&= \langle u \otimes v, j_* A_Y(c)\rangle \\
&= \langle j^* u \otimes j^* v, A_Y(c)\rangle \\
&= \langle j^* u \smile_i j^* v, c\rangle
\end{aligned}$$

$\square$

If $u, v \in S^*(X, A)$, then $j^*(q^*u \smile_i q^*v) = j^*q^*u \smile_i j^*q^*v = 0$. Therefore we have, $q^*u \smile_i q^*v \in \operatorname{im}(q^*)$ and q^* is 1-1 as a chain map. Thus, there is a unique cochain $u \smile_i v$ in $S^*(X, A)$ such that $q^*(u \smile_i v) = q^*u \smile_i q^*v$. Therefore the relative version $Sq^i : \mathrm{H}^p(X, A; \mathbb{Z}_2) \to \mathrm{H}^{p+i}(X, A; \mathbb{Z}_2)$ is defined by $q^*Sq^i = Sq^iq^*$.

Proposition 2.23. *1. If $A \subset X \subset (X, A)$, Sq^i commutes with $\delta^* : H^*(A) \to H^*(X, A)$.*

2. Sq^i commutes with suspension, i.e. $Sq^i \circ S^* = S^* \circ Sq^i$

$$\begin{tikzcd} \widetilde{H}^p(X) \arrow[r, "\delta^*"] \arrow[rr, bend right, "S^*"'] & H^{p+1}(CX, A) \arrow[r, "\cong"] & H^{p+1}(SX) \end{tikzcd}$$

Proof. Recall $\mathrm{H}^p(A) \xrightarrow{\delta^*} \mathrm{H}^{p+1}(X, A)$ is the map defined by $\alpha = \{a_p\} \mapsto \{c\}$, where

$$\begin{tikzcd} 0 \arrow[r] & S^p(X,A) \arrow[r] & S^p(X) \arrow[r] & S^p(A) \arrow[r] & 0 \\ & c = \{c_{p+1}\} \arrow[u, phantom, "\cup"] & b \arrow[u, phantom, "\cup"] \arrow[r, dashed, "j^*"] \arrow[d, "\delta"] & a = a_p \arrow[u, phantom, "\cup"] \arrow[d, "\delta"] & \\ & \delta^*(a_p) \sim c \arrow[r, dashed, "q^*"] & \delta^*(b) \arrow[r, dashed] & 0 & \end{tikzcd}$$

We need to verify $Sq^i(\delta^*\alpha) = Sq^i(c) = c \smile_{p-i+1} c \stackrel{?}{=} \delta^*(a \smile_{p-i} a) = \delta^*Sq^i(\alpha)$

But by definition $q^*(c \smile_{p-i+i} c) = q^*c \smile_{p-i+1} q^*c = \delta b \smile_{p-i+1} \delta b$. Now we claim the last expression $\delta b \smile_{p-i+1} \delta b = \delta b'$, where $b' = b \smile_{p-i+1} \delta b + b \smile_{p-i} b$ and therefore $j^*(b') = a \smile_{p-i} a$ (since $j^*(b) = a$ and $\delta(a) = 0$) $\implies \delta^*Sq^i(a) = Sq^i(\delta^*a)$.

It remains to check the claim. For this we use the formula of Lemma 2.17

$$\begin{aligned}
\delta(b') &= \delta(b \smile_{p-i+1} \delta b + b \smile_{p-i} b) \\
&= \delta(b \smile_{p-i+1} \delta b) + \delta(b \smile_{p-i} b) \\
&= \delta b \smile_{p-i+1} \delta b + b \smile_{p-i} \delta b + \delta b \smile_{p-i} b \\
&\quad + \delta b \smile_{p-i} b + b \smile_{p-i} \delta b + b \smile_{p-i-1} b + b \smile_{p-i-1} b \\
&= \delta b \smile_{p-i+1} \delta b \mod 2
\end{aligned}$$

To prove 2, we write the suspensin homomorphism S^* as the following composition:

$$\begin{tikzcd}
\alpha \in \widetilde{H}^q(X) \arrow[r, "\delta^*"] \arrow[rr, bend right, "S^*"'] & \mathrm{H}^{p+1}(CX, X) \arrow[r, "\cong", "\theta"'] & \mathrm{H}^{p+1}(SX)
\end{tikzcd}$$

$$\begin{aligned}
S^* \circ Sq^i(\alpha) &= \theta \circ \delta^*(Sq^i(\alpha)) \\
&= \theta Sq^i(\delta^* \alpha) \\
&= Sq^i(\theta \delta^* \alpha) \\
&= Sq^i \circ S^*(\alpha)
\end{aligned}$$

$\square$

Proposition 2.24. *Let $\delta_2 : H^p(X; \mathbb{Z}_2) \to H^{p+1}(X; \mathbb{Z}_2)$ be the Bokstein homomorphism corresponding to the exact sequence $0 \to \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$, then*

$$\delta_2 Sq^j = \begin{cases} 0 & \textit{if } j \textit{ odd} \\ Sq^{j+1} & \textit{if } j \textit{ even} \end{cases}$$

Proof. From Lemma 2.16 we induce the following identity, where $\varepsilon_j = 1 + (-1)^j$.

$$\delta(u \smile_{p-j} u) = \delta u \smile_{p-j} u + (-1)^p u \smile_{p-j} \delta u + (-1)^{p+1} \varepsilon_j u \smile_{p-j-1} u \tag{2.2}$$

We know the Bokstein $\delta_2 = [\beta]_2$ is the mod 2 of the Bokstein β of the exact sequence

$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. Now let $\mathbf{c} = \{c\} \in \mathrm{H}^p(X;\mathbb{Z}_2)$, $\mathbf{c} \in C^p(X;\mathbb{Z}_2)$. Pick $u \in C^p(X;\mathbb{Z})$ with $[u]_2 = c$. Then $Sq^j c = [u \smile_{p-j} u]_2$. Now by chasing diagrams we get

$$\begin{tikzcd}
 & & u \arrow[r, mapsto] \arrow[d, "\delta"'] & c \arrow[d, "\delta"'] & \\
C^{p+1}(X;\mathbb{Z}) \ni a \arrow[rr, mapsto, dashed, "\times 2"] & & 2a \equiv \delta u \arrow[r, mapsto, dashed] & 0 & \arrow[l] \text{means } \delta c \text{ is even}
\end{tikzcd}$$

$$\begin{aligned}
\text{But} \quad \delta_2 Sq^j(c) &= \frac{1}{2}\,\delta(u \smile_{p-j} u) \mod 2 \quad \text{then by using (2.2) and } \delta(u) = 0 \implies \\
&= \frac{\varepsilon_j}{2}\,(u \smile_{p-j-1} u) \mod 2 = \frac{\varepsilon_j}{2}\, Sq^{j+1}(c)
\end{aligned}$$

$\square$

Corollary 2.25. *$Sq^0(x) = x$, and $Sq^1(x) = \delta_2(x)$*

Proof. By above $\delta_2(Sq^0 u) = Sq^1(u)$. Hence in $\mathbb{R}\mathrm{P}^2$, $\delta_2(Sq^0(u_1)) = Sq^1(u_1) = u_1^2 \neq 0$ which implies $Sq^0(u_1) \neq 0$. Then since $\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) = \mathbb{Z}_2$ we get $Sq^0 = Id$ on $\mathbb{R}\mathbb{P}^2$.

The same can be proven for $\mathbb{R}\mathrm{P}^\infty$ by considering the exact sequence of the pair (Y, X), where $X = \mathbb{R}\mathrm{P}^2$ and $Y = \mathbb{R}\mathrm{P}^\infty$, and i is the inclusion $X \hookrightarrow Y$

$$\begin{tikzcd}
0 = \mathrm{H}^1(Y/X) \arrow[r] & \mathrm{H}^1(Y;\mathbb{Z}_2) \arrow[r, "i^*", "\cong"'] \arrow[d, "Sq^1"'] & \mathrm{H}^1(X;\mathbb{Z}_2) \arrow[r, "\delta"] \arrow[d, "Sq^1"', "\approx"] & \mathrm{H}^2(Y/X) = 0 \\
0 \arrow[r] & \mathrm{H}^2(Y;\mathbb{Z}_2) \arrow[r, "i^*", "\cong"'] & \mathrm{H}^2(Y;\mathbb{Z}_2) &
\end{tikzcd}$$

More generally for any X and $\alpha \in \mathrm{H}^1(X;\mathbb{Z}_2) = [X, \mathbb{R}\mathrm{P}^\infty]$, we can write $\alpha = f^*(u)$, for some map $f : X \to \mathbb{R}\mathrm{P}^\infty$, $\mathrm{H}^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2\langle u\rangle$.

$$Sq^1(\alpha) = Sq^1(f^* u) = f^*(Sq^1 u) = f^*(u \smile u) = f^*(u) \smile f^*(u) = u^2$$

Proposition 2.26 (Cartan Formula)**.** $\quad Sq^i(x \smile y) = \sum_j Sq^j(x) \smile Sq^{i-j}(y)$

Proof. Since $Sq^i(x \smile y) = Sq^i(d^*(x \times y)) = d^* Sq^i(x \times y)$, we first compute

$$\begin{aligned} Sq^i(x \times y) &= Sq^i((x \times 1) \smile (1 \times y)) \\ &= \sum_j Sq^j(x \times 1) \smile Sq^{i-j}(1 \times y) \\ &= \sum_j Sq^j(p^*(x)) \smile Sq^{i-j}(q^*(y)) \\ &= \sum_j p^* Sq^j(x) \smile q^* Sq^{i-j}(y) \\ &= \sum_j (Sq^j(x) \times 1) \smile (1 \times Sq^{i-j}(y)) \\ &= \sum_j Sq^j(x) \times Sq^{i-j}(y) \quad \Longrightarrow \end{aligned}$$

$$Sq^i(x \smile y) = d^*(\sum_j Sq^j(x) \times Sq^{i-j}(y)) = \sum_j Sq^j(x) \smile Sq^{i-j}(y)$$

□

Proposition 2.27. *There is an $\mathbb{Z}_2\langle T\rangle$-equivariant chain map $r : W \to W \otimes W$, where $W = S_*(S^\infty) = \langle e_j, Te_j \mid \partial e_j = e_{j-1} + Te_{j-1}\rangle$, and $T(u \otimes v) = Tu \otimes Tv$.*

Proof.

$$\begin{aligned} r(e_i) &= \sum (-1)^{j(i-j)} e_j \otimes T^j e_{i-j} = \sum_{p+q=i} (-1)^{pq} e_p \otimes T^p e_q \\ r(Te_i) &= T(r(e_i)) \end{aligned}$$

Recall the chain map $\varphi : W \otimes S_*(X) \to S_*(X) \otimes S_*(X)$. Set $K = S_*(X)$ and write

$$\begin{gathered} \varphi_K : W \otimes K \to K \otimes K \\ e_j \otimes \sigma \mapsto A_j(\sigma) \\ Te_j \otimes \sigma \mapsto CA_j(\sigma) \end{gathered}$$

Then by Section 2.1 the following diagram commutes up to chain equivalence.

$$\begin{tikzcd}
W \otimes K \otimes L \arrow[r, "r \otimes Id"] \arrow[dr, "\varphi_{K \otimes L}"'] & (W \otimes W) \otimes K \otimes L \arrow[r, "T", "\text{shuffle}"'] & (W \otimes K) \otimes (W \otimes L) \arrow[d, "\varphi_K \otimes \varphi_L"'] \\
& (K \otimes L) \otimes (K \otimes L) & (K \otimes K) \otimes (L \otimes L) \arrow[l, "T"', "\text{shuffle}"]
\end{tikzcd}$$

$$\begin{tikzcd}
e_i \otimes (a \otimes b) \arrow[r] \arrow[ddr, "\varphi_{K \otimes L}"'] & (\sum_j \pm e_j \otimes T^j(e_{i-j})) \otimes (a \otimes b) \arrow[d] \\
& \sum_j \pm (e_j \otimes a) \otimes (T^j(e_{i-j}) \otimes b) \arrow[d] \\
& \sum_j A_j(a) \otimes C^j A_{i-j}(b)
\end{tikzcd}$$

$$\begin{aligned}
\therefore \quad Sq^i(u \otimes v)(a \otimes b) &= ((u \otimes v) \smile_i (u \otimes v))\,(a \otimes b) \\
&= ((u \otimes v) \otimes (u \otimes v))\,\varphi_{K \otimes L}(e_n \otimes a \otimes b) \\
&= ((u \otimes v) \otimes (u \otimes v)) \sum A_j(a) \otimes C^j A_{i-j}(b) \\
&= \sum (u \smile_j u)(a) \otimes (v \smile_{i-j} v)(b) \\
&= \sum (Sq^j(u) \times Sq^{i-j}(v))(a \otimes b)
\end{aligned}$$

$\square$

Proposition 2.28 (Adem relations)**.** *For $a < 2b$,*

$$Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

If we write $Sq = Sq^0 + Sq^1 + \cdots : \mathrm{H}^*(X;\mathbb{Z}_2) \to \mathrm{H}^*(X;\mathbb{Z}_2)$ Cartan formula becomes

$$Sq(xy) = Sq(x)Sq(y).$$

$$\begin{aligned} \text{If } x \in \mathrm{H}^1(X) \implies Sq(x^k) &= (Sq(x))^k \\ &= (x+x^2)^k \\ &= x^k(1+x)^k \\ &= \sum_i \binom{k}{i} x^{k+i} \end{aligned}$$

Therefore, $Sq^i(x^k) = \binom{k}{i}x^{k+i}$.

Example 2.3. *Let $f : P^n \to P^n/P^r$ be collapse map. As a CW-complex:*

$$P^r \hookrightarrow P^n \underset{f}{\hookrightarrow} P^n/P^r = e^{r+1} \cup \cdots \cup e^n$$

$$H^{k-1}(P^r) \to H^k(P^n/P^r) \underset{\cong}{\xrightarrow{f^*}} H^k(P^n) \quad \forall\ k > r$$

Let $w_k \in H^k(P^n/P^r)$ with image $u^k \in H^k(P^n)$. Then, $Sq^i(w_k) = \binom{k}{i}w_{k+i}$. In particular, for $n = 5$, $r = 2$, $k = 3$ $Sq^2(w_3) = w_5$ (not related to cup product). We can use this to show P^4/P^2 not a retract of P^5/P^2. Because if there was a retraction map r

$$\begin{tikzcd} P^4/P^2 \arrow[r, bend right, "i"'] & P^5/P^2 \arrow[l, bend right, "r"'] \end{tikzcd}$$

with $i \circ r \simeq Id$, and $r \circ i = Id$, the following diagram gives a contradiction.

$$\begin{tikzcd} & H^3(P^4/P^2) \arrow[r, "r^*"] \arrow[d, "Sq^2"'] & H^3(P^5/P^2) \arrow[d, "Sq^2"] & \ni\ w_2 \arrow[d, dashed] \\ 0 \in & H^5(P^4/P^2) \arrow[r, "r^*"] & H^5(P^5/P^2) & \ni\ w_5 \end{tikzcd}$$

Lemma 2.29. *If* $a = \sum a_j 2^j$, $b = \sum b_j 2^j$ *(a_j, b_j are either 0 or 1) then*

$$\binom{a}{b} = \prod \binom{a_j}{b_j} \mod 2$$

where $\binom{0}{1} = 0$, $\binom{1}{0} = \binom{1}{1} = \binom{0}{0} = 1$.

Proof.

$$\begin{aligned} (1+x)^a &= (1+x)^{\sum a_j 2^j} = \prod (1+x)^{a_j 2^j} = \prod_j (1+x^{2^j})^{a_j} \\ &= \prod_j \sum_k \binom{a_j}{k} x^{k2^j} = \prod_j \left(1 + \binom{a_j}{1} x^{2^j} + \binom{a_j}{2} x^{2(2^j)} + \dots\right) \end{aligned}$$

The coefficient of x^b on the LHS $= \binom{a}{b}$ and the coefficient of x^b on the RHS $= \prod \binom{a_j}{b_j}$. Therefore $\binom{a}{b} = 1 \iff (b_j = 1 \implies a_j = 1)$. $\square$

Since when $\deg(x) = 1$, $Sq^b(x^{2^k}) = \binom{2^k}{b} x^{2^k+i}$ from the lemma we conclude

$$Sq^i(x^{2^k}) = \begin{cases} x^{2^k} & \text{if } i = 0 \\ x^{2^{k+1}} & \text{if } i = 2^k \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.30. Sq^i *is indecomposable if and only if* $i = 2^N$.

Proof. For ($\Longleftarrow$) take $i = 2^N$. Consider $\mathrm{H}^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2\langle\alpha\rangle$

$$Sq(\alpha^i) = (Sq(\alpha))^i = (\alpha + \alpha^2)^i = \alpha^i + \alpha^{2i}$$

Thus, $Sq^t(\alpha^i) = 0$ unless $t = 0$ or i. Therefore, $\alpha^{2i} \neq 0$ shows Sq^i is indecomposable. Otherwise, $\alpha^{2i} = Sq^i(\alpha^i) = \sum_{t<i} a_t Sq^t(\alpha^i) = 0$. To prove the converse ($\Longrightarrow$) assume

$i \neq 2^N$. Write $i = a + 2^k$ with $0 < a < 2^k$ and $b = 2^k$. Adem relations imply

$$Sq^a Sq^b = \binom{b-1}{a} Sq^{a+b} + \sum_{c>0} \binom{b-c-1}{a-2c} Sq^{a+b-1} Sq^c = Sq^{a+b} + \dots$$

This is because we calculate the first coefficient to be 1 (since all of $b_j = 1$).

$$\binom{b-1}{a} = \binom{2^k - 1}{a} = \binom{1 + 2 + 2^2 + \cdots + 2^{k-1}}{\sum a_i 2^i} = \prod \binom{b_i}{a_i} = 1 \mod 2$$

□

2.6.2 Hopf Invariant

For $n \geq 2$, let $f : S^{2n-1} \to S^n$ be a map and Let $K = S^n \cup_f e^{2n}$. Then

$$\mathrm{H}^i(K) = \begin{cases} 0 & \text{if } i \neq 0, n, 2n \\ \mathbb{Z} & \text{if } i = 0, n, 2n \end{cases}$$

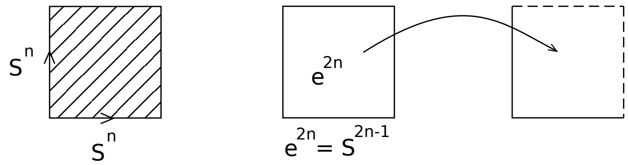
Let $\mathrm{H}^n(K) = \mathbb{Z}\langle\sigma\rangle, \mathrm{H}^{2n}(K) = \mathbb{Z}\langle\tau\rangle$. Then $\sigma^2 = H(f)\tau$. The integer $H(f)$ is called the Hopf invariant, it is a map $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ Next, we list some properties:

1. If n is odd, then $H(f) = 0$.
2. If $n = 2, 4, 8$, there is a map $f : S^{2n-1} \to S^n$ with Hopf invariant 1.

The first property follows from cup product being skew symmetric, the second follows from the cup product structures of various projective spaces: For $n = 2$, take $f : S^3 \to \mathbb{C}\mathrm{P}^1 = S^2$, then $\mathbb{C}\mathrm{P}^2 = S^2 \cup_f e^4$. Similarly for $n = 4$ and 8 we take $\mathbb{H}\mathrm{P}^2 = S^4 \cup e^8$ and Cay $\mathrm{P}^2 = S^8 \cup e^{16}$. All of the attaching maps have Hopf invariant 1.

Proposition 2.31. *If n is even, there is a map $f : S^{2n-1} \to S^n$ with $H(f) = 2$.*

Proof. For this write $S^n \times S^n = (S^n \vee S^n) \cup_g e^{2n}$, where $g : \partial e^{2n} \to S^n$ is the attaching map. Let $\varphi = F \circ g$ be the composition of g and the folding map F. We claim $H(\varphi) = 2$.



$$
\begin{tikzcd}
e^{2n} \supset S^{2n-1} \arrow[r, "g"] \arrow[rr, bend left, "\varphi"] & S^n \vee S^n \arrow[r, "F"] \arrow[d, "\mathrm{inc}"] & S^n \arrow[d, "i"] \\
S^n \times S^n = & (S^n \vee S^n) \cup_g e^{2n} \arrow[r, dashed, "\bar{F}"] & S^n \cup_\varphi e^{2n} = K
\end{tikzcd}
$$

Let $K = S^n \cup_\varphi e^{2n}$ and $i : S^n \to K$ be the inclusion. The composition $i \circ F \circ g : S^{2n-1} \to K$ is null homotopic by definition, so we can exted it to a map $\bar{F} : S^2 \times S^2 \to K$, which is homeomorphisim on the top cell e^{2n}. Hence $\bar{F}^*$ is isomorphism in dimension $2n$.

Make identifications $\mathrm{H}^n(S^n) = \mathbb{Z}\langle\sigma\rangle \cong H^n(K)$ and $\mathrm{H}^n(S^n \vee S^n) = \mathbb{Z}\langle\sigma_1, \sigma_2\rangle \cong H^2(S^2 \times S^2)$. The folding map has the property $F^*(\sigma) = \sigma_1 + \sigma_2$. therefore $\sigma^2 = H(\varphi)\tau$, Then $(\bar{F}^*(\sigma))^2 = \bar{F}^*(\sigma^2) = H(\varphi)\bar{F}^*(\tau) = H(\varphi)\rho$. Then the following implies $H(\varphi) = 2$.

$$(\bar{F}^*(\sigma))^2 = (F^*\sigma)^2 = (\sigma_1 + \sigma_2)^2 = \sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2 + \sigma_2\sigma_1 = \rho + (-1)^{n^2}\rho = 2\rho$$

□

Also it is easy to check that the Hopf invariant $H : \pi_{2n-1}(S^n) \to S^n$ is a homomorphism. This is because $H(f+g)$ can be computed from from K, which is obtained by attaching e^{2n} to S^n via $F \circ (f \vee g) \circ Q$, where Q is the pinch map $S^{2n} \to S^{2n} \vee S^{2n}$.

Theorem 2.32. *If there is a Hopf invariant 1 map $f : S^{2n-1} \to S^n$, then $n = 2^k$.*

Proof. If there is such a map $Sq^n(\sigma^n) = \tau$ in $K = S^n \cup_f e^{2n}$, where σ, τ are generators of $\mathrm{H}^n, \mathrm{H}^{2n}$. If $n \neq 2^k$, it is decomposable since Sq^n is decomposable where $Sq^i(\sigma) = 0$ for all $0 < i < n$. □

Theorem 2.30 implies that if $i \neq 2^N$ and $\mathrm{H}^k(X;\mathbb{Z}_2) = 0$ for all $n < k < n+i$, then

$$Sq^i : \mathrm{H}^n(X;\mathbb{Z}_2) \to \mathrm{H}^{n+i}(X;\mathbb{Z}_2)$$

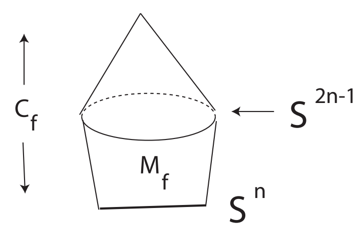
is zero. If $n \neq 2^N$, then Sq^n is decomposable. Clearly if $x \in \mathrm{H}^n(X;\mathbb{Z}_2)$ and $x^2 \neq 0$, then $Sq^{2^i}(x) \neq 0$ for some $0 < 2^i \leq n$.

Corollary 2.33. *(a) If $H^*(X;\mathbb{Z}_2) = \mathbb{Z}_2[x]$ or $\mathbb{Z}_2[x]/(x^q)$ for some $q \geq 2$ then $n = \deg(x)$ is a power of 2. (Adams $\implies$ $n = 1, 2, 4$ or 8.)*

(b) If M^{2n} is an $(n-1)$-connected closed manifold $\implies$ $n = 2^N$.

(c) If $S^{n-1} \to S^{2n-1} \underset{f}{\to} S^n$ fiber bundle then $n = 2^N$ (Adams $\implies$ $n = 1, 2, 4$ or 8.)

Proof. (a) and (b) follows from the above arguments. For (c) we let M_f to be the mapping cylinder of f, which is a $2n$-manifold whose boundary is S^{2n-1}. Then mapping cone C_f is a closed $2n$-manifold with $\mathrm{H}_i(C_f;\mathbb{Z}_2) = 0$ for $i < n$ and $\mathrm{H}_n(C_f;\mathbb{Z}_2) \cong \mathbb{Z}_2$. $\square$



To sum up $\quad Sq^i : \mathrm{H}^n(X;\mathbb{Z}_2) \to \mathrm{H}^{n+i}(X;\mathbb{Z}_2)$

1. Sq^i is an additive homomorphism and has naturality.
2. $Sq^0 = Id$
3. $Sq^i(\alpha_i) = \alpha_i^2$
4. $Sq^i(\alpha_n) = 0$ for $i > n$
5. $Sq^i(x \smile y) = \sum_{i+j=n}(Sq^i x) \smile (Sq^j y)$

These axioms imply $Sq^1 = \delta_2$, Adem relations, and $S^*Sq^i = Sq^iS^*$.

There is also the *Steenrod p-th power* operations generalizing Steenrod squares.

$$P^i : \mathrm{H}^n(X; \mathbb{Z}_p) \to \mathrm{H}^{n+2i(p-1)}(X; \mathbb{Z}_p) \text{ for } p > 2$$

1. P^i is an additive homomorphism and has naturality.
2. $P^0 = Id$
3. $P^i(\alpha_{2i}) = \alpha_{2i}^p$
4. $P^i(\alpha_n) = 0$ for all $2i > n$
5. $P^n(x \smile y) = \sum_{i+j=n} P^i x \smile P^j y$

These axioms imply Adem relations and $S^*P^i = P^iS^*$.

Remark 2.34.

- Every $\alpha \in \mathrm{H}_k(X; \mathbb{Z})$ is representable as image of an orientable k-manifold for $k \leq 6$
- Every $\alpha \in \mathrm{H}_k(X; \mathbb{Z}_2)$ is representable by images of nonorientable k-manifolds for nonorientable X.
- For every $k \geq 7$ there is a polyhedra X and $\alpha \in \mathrm{H}_k(X)$ which cannot be representable as images of orientable k-manifolds.

Chapter 3

Duality and Intersection theory

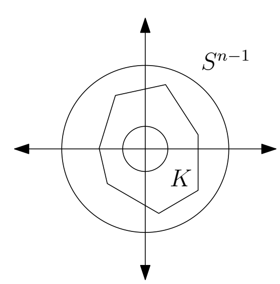
3.1 Poincaré Duality

Lemma 3.1. *Let M^m be m-manifold without boundary, and $K \subset M$ compact, then*

- $H_i(M, M-x) = 0 \ \ \forall i > m$
- $\alpha \in H_m(M, M-K) = 0 \iff \forall x \in K$ *the following holds (ρ_x is the restriction)*

$$\begin{array}{ccc} H_m(M, M-K) & \xrightarrow{\rho_x} & H_m(M, M-x) \\ \cup & & \cup \\ \alpha & \longrightarrow & \rho_x(\alpha) = 0, \end{array}$$

Proof. First, if $M = \mathbb{R}^m$, K is compact convex, this holds since there are deformations retractions $\mathbb{R}^n - 0 \to S^{n-1} \leftarrow \mathbb{R}^n - K$ given by $r_t(x) = t(x/|x\|) + (1-t)x$ inducing an isomorphism $\mathrm{H}_m(\mathbb{R}^m, \mathbb{R}^m - K) \cong \mathrm{H}_m(\mathbb{R}^m, \mathbb{R}^m - x)$.



Now we claim that if the Lemma holds for K_1, K_2, and $K_1 \cap K_2$ then it holds for $K = K_1 \cup K_2$. To see this first recall the chain homotopy $S_*(U_1) + S_*(U_2) \simeq S_*(U_1 \cup U_2)$ given by excision, which implies the exact sequence (3.1), which in turn induces the long exact sequence in homologies below. The maps in the long exact sequence are competible with the restriction maps ρ_x, this implies the result.

$$0 \to S_*(M, U_1 \cap U_2) \to S_*(M, U_1) \oplus S_*(M, U_2) \to S_*(M, U_1 \cup U_2) \to 0 \tag{3.1}$$

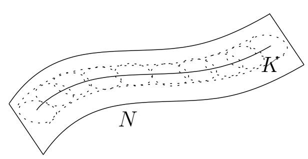
$$\to \mathrm{H}_{i+1}(M, M - K_1 \cap K_2) \xrightarrow{\delta} \mathrm{H}_i(M, M - K) \xrightarrow{s} \mathrm{H}_i(M, M - K_1) \oplus \mathrm{H}_i(M, M - K_2) \to$$

By repeated application of this, we see that the Lemma holds for $K \subset \mathbb{R}^m$, where $K = K_1 \cup \cdots \cup K_m$, such that each K_i is a compact convex subset of $\mathbb{R}^m$.

In case $K \subset \mathbb{R}^m$ (a chart), and $a \in \mathrm{H}_i(M, M - K)$, we can choose a compact neighborhood N of K with $int(N) \neq \phi$ such that there is a lifting of the chain a to a'

$$\begin{array}{ccc}
\mathrm{H}_i(\mathbb{R}^m, \mathbb{R}^m - N) & \xrightarrow{\rho_K} & \mathrm{H}_i(\mathbb{R}^m, \mathbb{R}^m - K) \\
\Cup & & \Cup \\
a' & \mapsto & a
\end{array}$$

Then we cover K by closed balls $\{B_i\}$, with $K \subset \cup_i B_i \subset N$. For simplicity call $B = \cup_i B_i$



Since Lemma holds for B, and lifting a' factors through a'' the lemma holds for K

$$\begin{tikzcd}
a' \in \mathrm{H}_i(\mathbb{R}^m, \mathbb{R}^m - N) \arrow[r, "\rho_B"] \arrow[d] & \mathrm{H}_i(\mathbb{R}^m, \mathbb{R}^m - B) \ni a'' \arrow[d] \arrow[dl, dashed] \\
\mathrm{H}_i(\mathbb{R}^m, \mathbb{R}^m - x) \arrow[r, leftrightarrow, "\rho_x"] & \mathrm{H}_i(\mathbb{R}^m, \mathbb{R}^m - K) \ni a
\end{tikzcd}$$

The most general case follows by writing K as a union of finetely many compact sets, each of which is lying in a chart, and appying the above argument. $\square$

Definition 3.2. A *local orientation* μ_x of a closed manifold M^m at x is a choice of one of two possible generators of $\mathrm{H}_m(M, M-x) \cong \mathbb{Z}$

Remark 3.3. Any such μ_x determines local orientation μ_y at all y near x. To see this, we take a ball containing both x and y and consider the isomorphisms:

$$\begin{tikzcd}[column sep=tiny] & \mathrm{H}_m(M, M-B) \arrow[dl] \arrow[dr] & \\ \mathrm{H}_m(M, M-x) & & \mathrm{H}_m(M, M-y) \end{tikzcd}$$

Definition 3.4. An *orientation* ϕ of M is a function which associates to each $x \in M$ a local orientation μ_x, which varies continuously with x. This last condition means that $\forall x \in M \;\; \exists$ compact neighborhood $x \in N$ and a class μ_N mapping to μ_x by restriction.

$$\begin{array}{ccc} \mathrm{H}_m(M, M-N) & \xrightarrow{\rho_N} & \mathrm{H}_m(M, M-x) \\ \Cup & & \Cup \\ \mu_N & \mapsto & x \end{array}$$

(M, ϕ) is called an oriented manifold.

Theorem 3.5. *For any oriented manifold $(M.\phi)$, and a compact set $K \subset M$, there exists a unique $\mu_K \in H_m(M, M-K)$ such that μ_K maps to μ_x by restriction:*

$$\begin{array}{ccc} H_m(M, M-K) & \to & H_m(M, M-x) \\ \Cup & & \Cup \\ \mu_K & \mapsto & \mu_x \end{array}$$

In particular if M is compact then $\mu_M \in H_m(M)$ represents ϕ.

Proof. Uniqueness follows from the previous lemma. First of all, if K is a sufficiently small compact set the result follows from the definition. And since we can write K as a union of small compact sets, it suffices to prove that if $K = K_1 \cup K_2$ and μ_{K_1} and μ_{K_2} exists $\implies$ μ_K exists. This fallows from the following exact sequence.

$$0 \to \mathrm{H}_m(M, M-K) \xrightarrow{s} \mathrm{H}_m(M, M-K_1) \oplus \mathrm{H}_m(M, M-K_2) \xrightarrow{t} \mathrm{H}_i(M, M-K_1 \cap K_2)$$

By uniqueness restrictions of μ_{K_j} $j = 1, 2$ maps $\mu_{K_1 \cap K_2}$, so $t(\mu_{K_1} \oplus \mu_{K_2}) = 0$. Hence there is some $\mu_K \in \mathrm{H}_m(M, M-K)$ with required properties (by uniqueness) □

Corollary 3.6. *If M is a closed oriented manifold, then there is a unique $\mu_M \in H_m(M^m) = H_m(M, M-M)$ such that $\rho_x(\mu_M) = \rho_x$. This is called a fundamental class of M.*

Remark 3.7. A similar proof gives the following: If M^m compact oriented with boundary, then for any compact $K \subset M$ there is a unique $\mu_K \in \mathrm{H}_m(M, (M-K) \cup \partial M)$ such that $\rho_x(\mu_K) = \mu_x$, for all $x \in K \cap (M - \partial M)$. By taking $K = M$ we get $\mu_M \in \mathrm{H}_m(M, \partial M)$.

Theorem 3.8 (Poincaré Duality)**.** *If M^m closed oriented then there is an isomorphism:*

$$\begin{array}{rcl} (a) \quad \Phi : H^i(M) & \xrightarrow{\cong} & H_{m-i}(M) \\ \Phi(a) & = & a \frown \mu_M \end{array}$$

If M is an open oriented manifold then we have the isomorphism:

$$(b) \quad \Phi : H_c^i(M) \xrightarrow{\cong} H_{m-i}(M)$$

where $H_c^i(M) = \varinjlim_K H^i(M, M-K)$, where the direct limit is taken over any nested sequence of compact sets filling M. The map Φ is induced from the cap product opeartion:

$$\begin{array}{rcl} H^i(M, M-K) \otimes H_m(M, M-K) & \to & H_{m-i}(M) \\ a \otimes \mu_K & \mapsto & a \frown \mu_K \end{array}$$

This map is defined by Proposition 2.10. Now we can take the direct limit, because for any compact pair $K \subset L$ we have commuting diagram (where i^* is induced from *inclusion):*

$$\begin{array}{ccc} H^i(M, M-K) & \xrightarrow{i^*} & H^i(M, M-L) \\ \downarrow \frown \mu_K & & \downarrow \frown \mu_L \\ H_{m-i}(M) & \xrightarrow{i_*} & H_{m-i}(M) \end{array}$$

$$i_*(i^*(a) \frown \mu_L) = a \frown i_*(\mu_L) = a \frown \mu_K$$

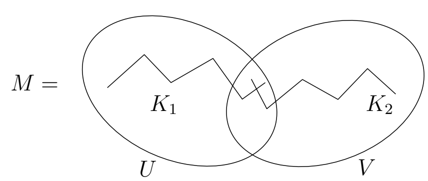
Proof. : We proceed in several steps. When $M = \mathbb{R}^m$, we can take balls of increasing radius $K = B^m$ as exhausting compact subsets of M, check the validity of the theorem $\mathrm{H}_i(\mathbb{R}^m, \mathbb{R}^m - B^m) = \mathrm{H}_i(B^m, \partial B^m) = 0$ or $\mathbb{Z}$ if $i = m$, or $i \neq m$. So we calculate $\mathrm{H}^m_c(\mathbb{R}^m) = \varinjlim_K \mathrm{H}^m(\mathbb{R}^m, \mathbb{R}^m - B) = \mathrm{Hom}(\mathrm{H}_m(\mathbb{R}^m, \mathbb{R}^m - B), \mathbb{Z}) = \mathbb{Z}\langle a \rangle \cong \mathbb{Z}\langle \mu_B \rangle$. Hence $1 = \langle a, \mu_B \rangle = \langle a \smile 1, \mu_B \rangle = \langle 1, a \frown \mu_B \rangle \implies a \frown \mu_B$ is the generator of $\mathrm{H}_0(\mathbb{R}^m)$.

Hence the map $\Phi : \mathrm{H}^m(\mathbb{R}^m, \mathbb{R}^m - B) \to \mathrm{H}_0(\mathbb{R}^m)$ $a \mapsto a \frown \mu_{\mathbb{R}}$ induces the isomorphism:

$$\begin{array}{rcl} \Phi : \mathrm{H}^i_c(\mathbb{R}^m) & \to & \mathrm{H}_{m-i}(\mathbb{R}^m) \\ a & \mapsto & a \frown \mu_B \end{array}$$

Next we show, if theorem holds for $U,\ V,\ U \cap V$ then it holds for the union $M = U \cup V$. If $K \subset U$, $L \subset V$ compact sets then the long excat sequence with excision gives

$$\begin{array}{ccccc} \xrightarrow{\delta} \mathrm{H}_i(M, M - K \cap L) & \to & \mathrm{H}_i(M, M-K) \oplus \mathrm{H}_i(M, M-L) & \to & \mathrm{H}_i(M, M - K \cup L) \to \\ \wr\!\parallel \text{exc} & & \wr\!\parallel \text{exc} & & \\ \mathrm{H}^i(U \cap V, U \cap V - K \cap L) & & \mathrm{H}^i(U, U-K) \oplus \mathrm{H}^i(V, V-L) & & \end{array}$$



Then by taking direct limits we get a commuting diagram, then 5-lemma gives the result:

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\delta} & \mathrm{H}^i_c(U \cap V) & \to & \mathrm{H}^i_c(U) \oplus \mathrm{H}^i_c(U) & \to & \mathrm{H}^i_c(U \cup V) & \xrightarrow{\delta} & \dots \\ & & \cong\downarrow \cap \mu_{K\cap L} & & \cong\downarrow \cap \mu_K \oplus \cap \mu_L & & \downarrow \cap \mu_M & & \\ & & \to \mathrm{H}_{m-i}(M) & \to & \mathrm{H}_{m-i}(U) \oplus \mathrm{H}_i(V) & \to & \mathrm{H}_{m-i}(M) \to & & \end{array}$$

So this means theorem holds for any open subset $U \subset \mathbb{R}^m$. This is because U is a union of open balls, hence it is an increasing union of open sets $U_1 \subset U_2 \subset \ldots U$, each of which is a finite union of balls (also note that any compact $K \subset M$ must be containen in some U_α). Then theorem holds by taking direct limits:

$$\begin{aligned} \mathrm{H}^i_c(M) &= \varinjlim \mathrm{H}^i_c(U_\alpha) \\ \mathrm{H}_{m-i}(M) &= \varinjlim \mathrm{H}_{m-i}(U_\alpha) \end{aligned}$$

Then the theorem holds for any smooth manifold M by taking direct limits over the open subsets, each of which is a finite union of charts. □

3.2 Lefschetz and Alexander Duality

Let M smooth compact oriented manifold with boundary ∂M. Then the boundary has a collared neighborhood, i.e. a neighborhood homeomorphic to $\partial M \times (0, 1]$. Then from the open version of Poincare Duality above, we get the following with boundary version, which is usually called Lefschetz duality.

$$H^k(M, \partial M) \cong H^k_c(M - \partial M) \cong H_{m-k}(M - \partial M) \cong H_{m-k}(M) \tag{3.2}$$

The isomorphism between the first and the last group can be seen to be simply capping with the class μ_M, where μ_M is obtained by first doubling M and getting a closed oriented manifold manifold $D(M) = M_+ \cup -M_-$ then taking the orientation class μ_{M_-}

$$\mu \in \mathrm{H}_{m-q}(M, \partial M) \cong \mathrm{H}_{m-q}(D(M), D(M) - M_-) \ni \mu_{M_-}$$

Theorem 3.9 (Lefschetz Duality). *Let M^m be a compact smooth oriented manifold such that $\partial M = A \cup B$ is a union of two codimension zero smooth manifolds A and B intersecting along their boundaries $A \cap B$ (which can be empty). Then capping with the orientation class $\mu = \mu_M \in H_m(M, \partial M)$ gives the following isomorphisms*

$$\frown \mu : H^k(M, A) \xrightarrow{\cong} H_{m-k}(M, B)$$

Proof. The inclusions $A \subset \partial M \subset M$ give rise the exact sequence of chain complexes

$$0 \to S_*(\partial M)/S_*(A) \to S_*(M)/S_*(A) \to S_*(M)/S_*(\partial M) \to 0$$

which gives the exact sequence in cohomology which is the top row of the following

$$\begin{array}{ccccccccc}
\cdots \to & H^k(M,\partial M) & \to & H^k(M,A) & \to & H^k(\partial M,A) & \to & H^{k+1}(M,\partial M) & \to \dots \\
& \downarrow\cong & & \downarrow\frown\mu & & \downarrow\cong & & \downarrow\cong & \\
\cdots \to & H_{m-k}(M) & \to & H_{m-k}(M,B) & \to & H_{m-k-1}(B) & \to & H_{m-k-1}(M) & \to \dots
\end{array}$$

The vertical maps are capping with μ except the third vertical map, which is the composition of the excision and capping with $\mu_B \in H_{m-1}(B,\partial B)$

$$H^k(\partial M,A) \cong H^k(B,\partial B) \xrightarrow{\frown\mu_B} H_{m-k-1}(B)$$

Then the version (3.2) and the 5-lemma implies the proof □

The following corollary is a special case of the Alexander duality theorem below.

Corollary 3.10. *If $K \subset S^n$ compact polyhedron, then $\forall j \neq n,\ n-1$ we have*

$$H^j(K) \xrightarrow{\cong} H_{n-j-1}(S^n - K) \tag{3.3}$$

Proof. Let U be a tubular neighborhood of K, and $C = S^n - U$ be its complement. Long exact sequence of the pair (S^n, K) gives isomorphism $\delta^* : H^j(K) \xrightarrow{\cong} H^{j+1}(S^n,K)$. Then the Poincare duality and excision gives the desired result:

$$H^{j+1}(S^n,K) \cong \mathrm{H}^{j+1}(S^n,U) \cong \mathrm{H}^{j+1}(C,\partial C) \cong \mathrm{H}_{n-j-1}(C) \cong \mathrm{H}_{n-j-1}(S^n-K)$$ □

Definition 3.11. Let $A \subset M$, then the check cohomology of A is the direct limit

$$\check{H}^q(A) = \varinjlim_V \mathrm{H}^q(V)$$

taken over the nested open sets $V_1 \supset V_2 \supset \dots$ with $\cap V_i = A$. So there is the natural restriction $\check{H}^q(A) \to H^q(A)$; which is an isomorphism when A is a polyhedron in M.

Theorem 3.12 (Alexander duality)**.** *Let M^m be a closed oriented manifold, and $K \subset M$ be a compcact subset. Then there is an isommorphism*

$$\check{H}^k(K) \cong H_{m-k}(M, M-K)$$

Proof. Let $\{V\}$ be decreasing collection of open set with $K \subset V$, and $\{C\}$ be increasing collection of compact sets with $V \subset C$, and $U = M - C$. Then we have the commuting diagram, where the vertical maps are capping with the fundamental class.

$$\begin{array}{ccccccccc}
\cdots \to & H^k(M, U \cup V) & \to & H^k(M, U) & \to & H^k(U \cup V, U) & \cong & H^k(V) & \to \dots \\
& \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow & \\
\cdots \to & H_{m-k}(M-K) & \to & H_{m-k}(M) & \to & H_{m-k}(M.M-K) & \cong & H_{m-k}(V, V-K) & \to \dots
\end{array}$$

and the top row is the long exact sequence induced from the short exact sequence:

$$0 \to S_*(U \cup V)/S_*(U) \to S_*(M)/S_*(U) \to S_*(M)/S_*(U \cup V) \to 0$$

Note that $U \cap V = \emptyset$. When we take the direct limits as C gets large and V small, the first two vertical maps become Poincare duality isomorphisms $H_c^k(M-K) \cong H_{m-k}(M-K)$ and $H_c^k(M) \cong H_{m-k}(M)$, and the third vertical map is the desired isomorphism (seen by using the 5-lemma). □

Remark 3.13. Suspension category is a category of polyhedra and continuous maps, where objects are defined up to suspension, i.e. $K \sim L$ if $\Sigma^k(K) \simeq \Sigma^k(L)$ for some k. We can extend this category by introducing virtual objects (K, n) where $\Sigma^n(K, n) = K$ in other words formally $(K, n) = \Sigma^{-n}(K)$. In this category we define the dual $\mathcal{D}(K)$ of K to be $(S^n - K, n-1)$ where $K \subset S^n$ is some imbedding (for $n \gg 1$). Then Alexander duality takes the pleasant form $H^j(K) \cong H_{-j}(\mathcal{D}(K))$, by following calculation

$$H^j(K) \cong H_{n-j-1}(S^n - K) \cong H_{n-j-1}(\Sigma^{n-1}(S^n - K, n-1)) \cong H_{-j}(\mathcal{D}(K))$$

3.3 Poincare duality via slant product

There is another useful characterization of the fundamental class as follows:

Lemma 3.14. *Let M^m be closed. Then, M is orientable if and only if there is a unique $U'_M \in H^m(M \times M, M \times M - \Delta)$ with $i_x^*(U'_M) = \mu_x$ generator of $H^m(M, M - x)$, where*

$$\begin{array}{ccc} (M, M - x) & \xrightarrow{i_x} & (M \times M, M \times M - \Delta) \\ y & \mapsto & (x, y) \end{array}$$

So there is $1-1$ correspondence between orientation μ of M and homology class U'_M

Proof. Let N be a tubular neighborhood of the diagonal $\Delta \subset M \times M$, then by excision

$$H^i(M \times M, M \times M - \Delta) \cong H^i(N, \partial N) \cong H_{2m-i}(N) = H_{2m-i}(M)$$

Then taking U'_M to be the cohomology corresponding class μ_M gives the proof. $\square$

Lemma 3.15. *Let $j : (M \times M, \phi) \to (M \times M, M \times M - \Delta)$, and $U_M = j^*(U'_M)$. Then $U_M/\mu_M = 1$ where*

$$\begin{array}{ccc} H^m(M \times M) \otimes H_m(M) & \to & H^0(M) \cong \mathbb{Z} \\ \Cup & & \Cup \\ U_M \otimes \mu_M & \to & U_M/\mu_M = 1 \end{array}$$

Proof. Consider the commutative diagram, where $i_x(y) = x \times y$

$$\begin{array}{ccc} (M, M - x) & \xrightarrow{i_x} & (M \times M, M \times M - \Delta) \\ \uparrow j_x & & \uparrow j \\ M & \xrightarrow{i_x} & M \times M \end{array}$$

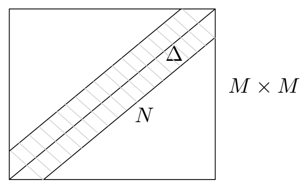
$$\begin{aligned} 1 &= \langle i_x^*(U'_M), \mu_x \rangle = \langle i_x^*(U'_M), (j_x)_* \mu_M \rangle = \langle j_x^* i_x^*(U'_M), \mu_M \rangle \\ &= \langle (i_x \circ j_x)^* U'_M, \mu_M \rangle = \langle (j \circ i_x)^*(U'_M), \mu_M \rangle = \langle i_x^*(U_M), \mu_M \rangle \\ &= \langle U_M, (i_x)_* \mu_M \rangle = \langle U_M, \bar{x} \times \mu_M \rangle = \langle U_M/\mu_M, \bar{x} \rangle \end{aligned}$$

$\square$

Lemma 3.16. $U_M \smile (\eta \times 1) = U_M \smile (1 \times \eta) \quad \forall \eta \in H^*(M)$

Proof. It suffices to check this on a tubular neighborhood N of the diagonal $\Delta \subset M \times M$. Let p_1 and p_2 are projections of $M \times M$ on the first and second factor respectively.

$$\begin{tikzcd} & & & M \\ \Delta & N \arrow[l, "r"'] \arrow[r, hook, "k"] & M \times M \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & & & M \end{tikzcd}$$



$$p_1|_N \cong p_2|_N \qquad \text{since} \quad i \circ r \simeq k \qquad \begin{tikzcd} N \arrow[rr, "r"] \arrow[dr, "k"'] & & \Delta \arrow[dl, hook', "i"] \\ & M \times M & \end{tikzcd}$$

$$\begin{array}{ccccccc} p_1|N & = & p_1 \circ k & \simeq & p_1 \circ (i \circ r) & = & (p_1 \circ i) \circ r \\ & & & & & & \shortparallel \\ p_2|N & = & p_2 \circ k & \simeq & p_2 \circ (i \circ r) & = & (p_2 \circ i) \circ r \end{array}$$

Also $\;p_1^*(a) = a \times 1\;$ and $\;p_2^*(a) = 1 \times a \implies k^*(a \times 1) = k^*(1 \times a)$

$$\begin{array}{ccc} \mathrm{H}^i(M \times M) & \xrightarrow{k^*} & \mathrm{H}^i(N) \\ \downarrow \smile U'_M & & \downarrow \smile k^* U'_M \\ \mathrm{H}^{i+m}(M \times M, M \times M - \Delta) & \underset{\cong}{\xrightarrow{k^*}} & \mathrm{H}^{i+m}(N, N - \Delta) \end{array}$$

$$\begin{array}{ccccc} k^*(U'_M \cup (\eta \times 1)) & = & k^*(U'_M \cup p_1^*(\eta)) & = & k^* U'_M \cup (p_1|_N)^*(\eta) \\ & & & & \shortparallel \\ k^*(U'_M \cup (\eta \times 1)) & = & k^*(U'_M \cup p_2^*(\eta)) & = & k^* U'_M \cup (p_2|_N)^*(\eta) \end{array}$$

$\square$

Theorem 3.17. *Let M^m be closed orientable $\Phi : H^p(M) \xrightarrow{\cong} H_{m-p}(M)$ Poincaré Duality, then*

$$\begin{gathered}\Phi^{-1} : H_{m-p}(M) \longrightarrow H^p\,(M)\\ \Phi^{-1}(a) = (-1)^{m(m-p)}\, U_M/a\end{gathered}$$

Proof. Let $\Psi(a) = U_M/a$, then by using Proposition we calculate

$$\begin{aligned}\Psi \circ \Phi(\alpha^p) &= \Psi(\alpha^p \frown \mu_M) = U_M/\alpha \frown \mu_M\\ &= 1^0 \smile [U_M/\alpha \frown \mu_M]\\ &= 1^{0\cdot m}[U_M \smile (1 \times \alpha)]/\mu_M\\ &= [U_M \smile (\alpha \times 1)]/\mu_M\\ &= (-1)^{pm}\alpha \smile [U_M/1 \frown \mu_M]\\ &= (-1)^{pm}\alpha \smile [U_M/\mu_M]\\ &= (-1)^{pm}\alpha \smile 1 = (-1)^{pm}\alpha\end{aligned}$$

□

3.4 Lefschetz fixed point theorem

Definition 3.18. We define the Lefschetz number of a map $f : M^m \longrightarrow M^m$ as:

$$\Lambda_f = \langle (f, id)^* U_M, \mu_M \rangle \in \mathbb{Q}\ , \ \text{ where } (f, id) \text{ is the composition}$$

$$M \xrightarrow{d} M \times M \xrightarrow{f \times id} M \times M$$

Proposition 3.19. $\wedge_f = \Sigma(-1)^p H_p(f)$

Proof. $\mathrm{H}^*(M \times M; Q) = \mathrm{H}^*(M; Q) \times\ \mathrm{H}^*(M \times Q)$. If we write in terms of generators $\mathrm{H}^*(M) = \langle \alpha_k \rangle$ where $\alpha_k \in H^k(M, Q)$, then $\mathrm{H}^*(M \times M; Q) = \langle \alpha_i \times \alpha_j \rangle$ and we can express $U_M = \Sigma c_{ij}(\alpha_i \times \alpha_j)$ and $f^*(\alpha_i) = \Sigma a_{ki}\alpha_k$. Then we calculate

$$\begin{aligned}
(-1)^{km}\alpha_k &= U_M/\alpha_k \frown \mu_M = \Psi \circ \Phi(\alpha_k) \\
&= \Sigma\, c_{ij}(\alpha_i \times \alpha_j)/\alpha_k \frown \mu_M = \Sigma\, c_{ij}\,\langle \alpha_j, \alpha_k \frown \mu_M \rangle\, \alpha_i \\
&= \sum_i \big(\sum_j c_{ij}\,\langle \alpha_j \smile \alpha_k, \mu_M \rangle\big)\alpha_i
\end{aligned}$$

$$\text{Therefore} \quad \sum_j c_{ij}\,\langle \alpha_j \smile \alpha_k, \mu_M \rangle = (-1)^{km}\delta_{ik} \implies$$

$$\begin{aligned}
\wedge_f &= \langle d^*(f \times id)^* U_M, \mu_M \rangle \\
&= \sum_{i,j} c_{ij}\,\langle d^*(f \times id)^*(\alpha_i \times \alpha_j), \mu_M \rangle \\
&= \sum_{i,j} c_{ij}\,\langle f^*\alpha_i \smile \alpha_j, \mu_M \rangle \\
&= \sum_{i,j,k} c_{ij} a_{ki}\,\langle \alpha_k \smile \alpha_j, \mu_M \rangle \\
&= \sum_{i,k} a_{ki}\big(\sum_j c_{ij}\,\langle \alpha_k \smile \alpha_j, \mu_M \rangle\big) \\
&= \sum_{i,k} a_{ki}(-1)^{km}\delta_{ik}(-1)^{k(m-k)} \\
&= \sum_i (-1)^i a_{ii} = \Sigma(-1)^i \text{ trace } \mathrm{H}_i(f)
\end{aligned}$$

□

3.5 Intersection Theory

Cup and cap products on manifolds can be computed geometrically by using Poincare duality. For example, the cup product of two cohomology classes α and β corresponds to the cohomology class whose dual is the intersection of the duals of α and β. Let M^m be a compact smooth oriented manifold. Recall that the Poincare duality is the isomorphism $D : \mathrm{H}_i(M^m) \xrightarrow{\cong} \mathrm{H}^{m-i}(M, \partial M)$ defined by $D^{-1}(\alpha) = \alpha \frown [M]$, where [M] is the fundamental class, so $D(a) \frown [M] = a$. When M^m is an open manifold then Poincare duality is defined similar way except using compactly supported cohomology

$$D : \mathrm{H}_i(M^m) \xrightarrow{\cong} \mathrm{H}_c^{m-i}(M)$$

Definition 3.20. Let $a, b \in H_*(M)$ we define the homology intersection $(a, b) \mapsto a.b$ by

$$\begin{aligned} a \cdot b &= D^{-1}(D(a) \smile D(b)) \\ &= (D(a) \smile D(b)) \frown [M] \\ &= D(a) \frown (D(b) \frown [M]) \\ &= D(a) \frown b \end{aligned}$$

We have $D(a \cdot b) = D(a) \smile D(b)$ which implies $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Definition 3.21. A smooth map between compact smooth manifolds $\pi : W^{n+k} \to N^n$ is called a *k-disk bundle* if there is an open covering $N = \bigcup_\alpha U_\alpha$ and there there are diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times D^k$ with (π_1 is the projection to first factor)

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha", "\cong"'] \arrow[dr, "\pi"'] & & U_\alpha \times D^k \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd}$$

such that $\phi_\alpha^{-1} \circ \phi_\beta : (U_\alpha \cap U_\beta) \times D^k \to (U_\alpha \cap U_\beta) \times D^k$ is given by $\phi_\alpha^{-1} \circ \phi_\beta(x, y) = (x, \theta_x(y))$ where $\theta : U_\alpha \cap U_\beta \to GL(k)$. In particular W is a manifold with boundary, and we have an inclusion map $N^n \xhookrightarrow{i} W^{n+k}$, satisfying $\pi \circ i = Id_N$, which is called the 0-section.

Definition 3.22. The *Thom class* of a k-disk bundle $\pi : W^{n+k} \to N^n$ over a closed manifold N is the cohomology class $\tau \in \mathrm{H}^k(W, \partial W)$ defined by $\tau = D_W(i_*[N])$.

Hence, $\tau \frown [W] = i_*[N]$, where $[W] \in \mathrm{H}_{n+k}(W, \partial W)$. By excision we have identification $\tau \in \mathrm{H}^k(W, \partial W) \cong \mathrm{H}^k(W, W - N)$. So we can write $\tau \in H_c^k(W)$ and $i^*(\tau) = \chi \in \mathrm{H}^k(N)$ is the **Euler class**. Thom class of a $\mathbb{R}^k$-vector bundle $W^{n+k} \xrightarrow{\pi} N^n$ is defined in the same way, in this case $\tau \in \mathrm{H}_c^k(W)$

Definition 3.23. If $f : N^n \to M^m$, we define the shriek maps $f^!$ and $f_!$ as follows

$$\begin{tikzcd} \mathrm{H}^{n-p}(N) \arrow[r, dashed, "f^!"] & \mathrm{H}^{m-p}(M) \\ \mathrm{H}_p(N) \arrow[u, "D_N", "\cong"'] \arrow[r, "f_*"] & \mathrm{H}_p(M) \arrow[u, "D_M", "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{H}_{m-p}(M) \arrow[r, dashed, "f_!"] & \mathrm{H}_{n-p}(N) \\ \mathrm{H}^p(M) \arrow[u, "D_M", "\cong"'] \arrow[r, "f^*"] & \mathrm{H}^p(N) \arrow[u, "D_N", "\cong"'] \end{tikzcd}$$

- $f^! = D_M f_* D_N^{-1}$, $f_! = D_N f^* D_M^{-1}$
- Similar definition when manifolds have boundaries.

Theorem 3.24 (Thom isomorphism)**.** *If $\pi : W^{n+k} \to N^n$ is a k-disk bundle, then there is an isomorphism $\Phi = i^! : H^p(N) \to H^{p+k}(W, \partial W)$ which is the composition:*

$$\Phi : H^p(N) \underset{\cong}{\xrightarrow{\pi^*}} H^p(W) \underset{\cong}{\xrightarrow{\smile\tau}} H^{p+k}(W, \partial W).$$

Also, $\Phi = i^! : H^p(N) \to H^{p+k}(W, \partial W)$ where $i : N \hookrightarrow W$ is the 0-section. Specificslly Φ can be expressed as the following map $\Phi(\alpha) = \pi^*(\alpha) \smile \tau$.

Proof. $i^!$ is isomorphism since i_* is isomorphism. Let $\beta = i^*(\alpha)$, so $\pi^*\beta = \alpha$, then

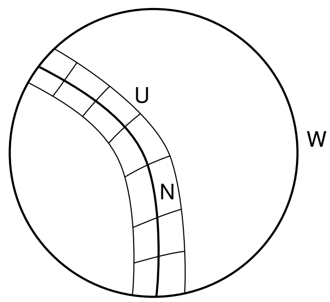
$$\begin{aligned} i^!(\beta) &= D_W i_* D_N^{-1}(\beta) \\ &= D_W i_*(i^*(\alpha) \frown [N]) \\ &= D_W(\alpha \frown i_*[N]) \\ &= D_W(\alpha \frown (\tau \frown [W])) \\ &= D_W((\alpha \smile \tau) \frown [W]) \\ &= \alpha \smile \tau \\ &= \pi^*(\beta) \smile \tau \end{aligned}$$

□

Under the following identifications, τ is the image of 1, and it is the unique class whose restriction to each fiber is the generator.

$$\mathbb{Z} \cong \mathrm{H}^0(N) \xrightarrow[\cong]{\Phi} \mathrm{H}^k(W, \partial W) \;\cong\; \mathbb{Z} \overset{\text{res}}{\dashrightarrow} \mathrm{H}^k(D^k, \partial) \cong \mathbb{Z},$$

Definition 3.25. Let $i_N^W : N^n \hookrightarrow W^m$ denote any proper inclusion.



We define the corresponding Thom class $\tau_N^W \in \mathrm{H}^{m-n}(W)$ by the correspondence

$$[N]_W = i^W_{N*}[N] \in \mathrm{H}_n(W, \partial W) \xrightarrow[\cong]{D_W} \mathrm{H}^{m-n}(W) \ni D_W([N]_W) = \tau_N^W$$

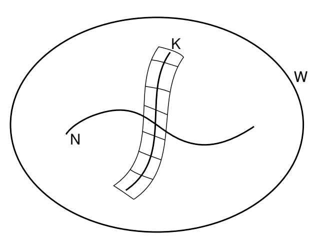
$$\begin{array}{ccc} \mathrm{H}^{m-n}(U, \partial U) \cong \mathrm{H}^{m-n}(W, W-U) & \xrightarrow{j^*} & \mathrm{H}^{m-n}(W) \\ \cup & & \cup \\ \tau & \dashrightarrow & \tau_N^W \end{array}$$

(Thom class)

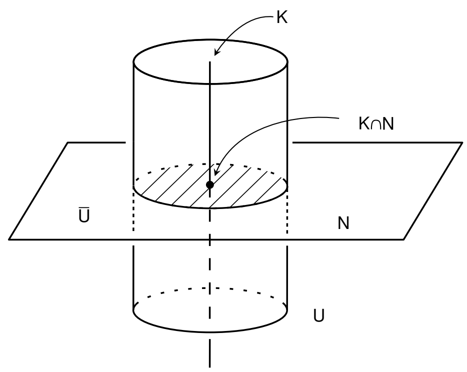
$$\tau_N^W \frown [W] = [N]_W = i_N^W[N] \tag{3.4}$$

3.5.1 Geometric intersections as duals of cup products

Let $K^k, N^n \hookrightarrow W^m$ closed smooth submanifolds, such that $K \pitchfork N$.



So we have the normal bundle identifications $\nu^N_{K\cap N} = \nu^W_K\big|_{K\cap N}$, so $\tau^N_{K\cap N} = (i^W_N)^*(\tau^W_K)$. Let $K \subset U$ be a tubular neighborhood of K, and $\bar{U} = U \cap N$.



$$\begin{tikzcd} K \cap N \arrow[r, hook] \arrow[d, hook] & N \arrow[d, hook] \\ K \arrow[r, hook] & W \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{H}^k(U, \partial U) \arrow[r] \arrow[d] & \mathrm{H}^k(\bar{U}, \partial \bar{U}) \arrow[d] \\ \tau^W_K \in \mathrm{H}^k(W) \arrow[r] & \mathrm{H}^k(N) \ni \tau^N_{K\cap N} \end{tikzcd}$$

Theorem 3.26. *If $K \pitchfork N$, then*

(a) $\tau^W_{K\cap N} = \tau^W_K \smile \tau^W_N$

(b) $[K \cap N]_W = [N]_W \cdot [K]_W$

$$\begin{tikzcd} K \cap N \arrow[r, hook, "i^N_{K\cap N}"] \arrow[rr, hook, bend right, "i^W_{K\cap N}"'] & N \arrow[r, hook, "i^W_N"] & W \end{tikzcd}$$

Proof.

$$\begin{aligned}
[K \cap N]_W &= (i^W_{K\cap N})_*[K \cap N] \\
&= (i^W_N)_*(i^N_{K\cap N})_*[K \cap N] \\
&= (i^W_N)_*(\tau^N_{K\cap N} \frown [N]) \\
&= (i^W_N)_*[(i^W_N)^*(\tau^W_K) \frown [N]] \\
&= \tau^W_K \frown \ (i^W_N)_*[N] \ \text{, then (3.4) and } \textit{Proposition } 2.9 \implies \\
&= (\tau^W_K \smile \tau^W_N) \frown [W] \\
&= [K]_W \cdot [N]_W
\end{aligned}$$

which proves (b). Also it implies that $\tau^W_{K\cap N} = \tau^W_K \smile \tau^W_N$ which is (a) $\square$

Corollary 3.27. *If M^m is closed,smooth orientable manifold, A, B open in M such that*

$$\begin{tikzcd}
\alpha \in H_p(A) \arrow[dr, dashed] \arrow[dr] & & H_q(B) \ni \beta \arrow[dl] \arrow[dl, dashed] \\
& \alpha_M \in H_*(M) \ni \beta_M &
\end{tikzcd}$$

then $\alpha_M \cdot \beta_M \neq 0$ implies $A \cap B \neq \emptyset$.

Proof.

$$\alpha \cdot \beta = (D_M(\alpha) \smile D_M(\beta)) \frown [M]$$

Notice also that $D_M(\alpha) \smile D_M(\beta) \in \mathrm{H}^{2m-p-q}(M, (M-A) \cup (M-B))$, then the proof follows from $(M-A) \cup (M-B) = M - A \cap B$. $\square$

3.6 Gysin Sequence

Let $E \xrightarrow{\bar{\pi}} X$ be an oriented k-disk bundle. Denote by $\dot{E} \xrightarrow{\pi} X$ the oriented $(k-1)$-sphere bundle induced from E. Gysin exact sequence is the following top horizontal sequence. The exactness follows from the lonf exact sequence of the pair $\dot{E} \subset E$ along with the proposition below.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^p(X) \arrow[r, "\smile \chi"] \arrow[d, "\Phi"', "\cong"] & \mathrm{H}^{p+k}(X) \arrow[r, "\pi^*"] \arrow[d, shift left, "\pi^*"] & \mathrm{H}^{p+k}(\dot{E}) \arrow[r, "\sigma^*"] \arrow[dr, "\delta^*"'] & \mathrm{H}^{p+1}(X) \arrow[r] \arrow[d, shift left, "\Phi", "\cong"'] & \cdots \\
& \mathrm{H}^{p+k}(E, \dot{E}) \arrow[r, "j^*"] & \mathrm{H}^{p+k}(E) \arrow[u, shift left, "i^*"] & & \mathrm{H}^{p+k+1}(E, \dot{E}) \arrow[u, shift left, "h"] &
\end{tikzcd}$$

Proposition 3.28. *(a) The following maps commute as $H^*(X)$-module homomorphisms, where Φ is the Thom isomorphism given by $\Phi(\alpha) = \bar{\pi}^*(\alpha) \smile \tau$*

$$\begin{tikzcd}
\beta \in H^p(X) \arrow[rr, "\smile \chi"] \arrow[d, "\Phi"', "\cong"] & & H^{p+k}(X) \arrow[d, shift left, "\bar{\pi}^*"] \\
H^{p+k}(E, \dot{E}) \arrow[rr, "j^*"] & & H^{p+k}(E) \arrow[u, shift left, "i^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\dot{E} \arrow[r, hook, "k"] \arrow[dr, "\pi"'] & E \arrow[r, "j"] \arrow[d, "\bar{\pi}"] & (E, \dot{E}) \\
& X &
\end{tikzcd}$$

$$\bar{\pi} \circ k = Id$$

(b) $\sigma^*(\pi^*\alpha \smile \beta) = (-1)^{\deg(\alpha)} \alpha \smile \sigma^*(\beta)$, where $\sigma^* = h \circ \delta^*$ and $h = \Phi^{-1}$.

Proof. Let us denote the inclusions $i : X \hookrightarrow E$ (as zero section), and $k : \dot{E} \hookrightarrow E$.

$$(a) \qquad i^* \circ j^*(\bar{\pi}^*(\beta) \smile \tau) \;=\; i^*(\bar{\pi}^*(\beta) \smile j^*\tau) = i^*\bar{\pi}^*(\beta) \smile i^*j^*\tau = \beta \smile \chi$$

To prove (b) set $i = \deg(\pi^*\alpha)$ and $j = \deg(\beta)$.

$$\begin{aligned}
\sigma^*(\pi^*\alpha \smile \beta) &= h\delta^*(\pi^*\alpha \smile \beta) = (-1)^{ij} h\delta^*(\beta \smile \pi^*\alpha) = (-1)^{ij} h\delta^*(\beta \smile k^*\bar{\pi}^*(\alpha)) \\
&= (-1)^{ij} h(\delta^*\beta \smile k^*\bar{\pi}^*(\alpha)) = (-1)^i h(\pi^*(\alpha) \smile \delta^*\beta) \\
&= (-1)^i \alpha \smile h\delta^*(\beta) = (-1)^i \alpha \smile \sigma^*(\beta)
\end{aligned}$$

In the next to last step we used $\Phi(\alpha \smile h\delta^*(\beta) = \pi^*(\alpha) \smile \delta^*\beta$. $\square$

Example 3.1. *If $S^{k-1} \longrightarrow \dot{E} \xrightarrow{\pi} X$ is a sphere bundle with section s, then $\chi = 0$.*

Proof. $\pi \circ s = Id$ implies $s^* \circ \pi^* = Id$. Since $\pi^*(\chi) = 0 \implies \chi = s^* \circ \pi^*(\chi) = 0$:

$$\begin{tikzcd}
\mathrm{H}^0(X) \arrow[r, "\smile \chi"] & \mathrm{H}^k(X) \arrow[r, "\pi^*"] & \mathrm{H}^k(\dot{E}) \arrow[r, "\sigma^*"] \arrow[l, bend left, "s^*"] & \mathrm{H}^0(X) \\
1 \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & \chi \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

$\square$

- If $S^{k-1} \to S^{m+k-1} \xrightarrow{\pi} M^m$ is a fibration with $k \geq 2$, then $m = kr$ for some r and

$$\mathrm{H}^*(M) \cong \mathbb{Z}[x]/(x^{r+1})$$

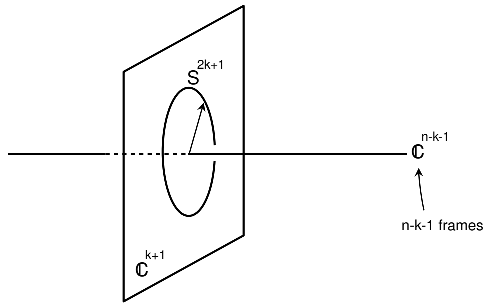
$$\mathrm{H}^{i+k-1}(S^{m+k-1}) \xrightarrow{\sigma^*} \mathrm{H}^i(M) \xrightarrow{\smile \chi} \mathrm{H}^{i+k}(M) \xrightarrow{\pi^*} \mathrm{H}^{i+k}(S^{m+k-1})$$

By homotopy exact sequence of fibration and Hurewicz theorem, the first nontrivial cohomology appears in $H^k(M) = \mathbb{Z}$ and is generated by χ (by Gysin sequence). By iterating this we get $1, \chi, \chi^2, \ldots, \chi^r$ as an additive basis for $\mathrm{H}^*(M)$.

Example 3.2. *Complex Steifel manifold of unitary k-frames in $\mathbb{C}^n$:*

$$\begin{aligned} V_{n,n-k} &= (n-k)\text{-}\mathit{frames\ in}\ \mathbb{C}^n \\ &\cong U(n)/U(k) \end{aligned}$$

$$\begin{tikzcd}
S^{2k+1} \arrow[r] \arrow[d, phantom, "\cong"] & V_{n,n-k} \arrow[r, "\pi"] \arrow[d, phantom, "\cong"] & V_{n,n-k-1} \arrow[d, phantom, "\cong"] \\
U(k+1)/U(k) \arrow[r, dashed] & U(n)/U(k) \arrow[r, dashed] & U(n)/U(k+1)
\end{tikzcd}$$



$$\begin{aligned} H^*(V_{n,n-k}) &= \Lambda(x_{2k+1}, x_{2k+3}, \ldots, x_{2n-1}) \text{ (exterior algebra)} \\ &= H^*(S^{2k+1} \times S^{2k+3} \times \cdots \times S^{2n-1}) \end{aligned}$$

Proof. We prove by induction on k. $\chi \in \mathrm{H}^{2k+2}(V_{n,n-k-1}) = 0$. By induction

$$0 \longrightarrow \mathrm{H}^j(V_{n,n-k-1}) \xrightarrow{\pi^*} \mathrm{H}^j(V_{n,n-k}) \xrightarrow{\sigma^*} \mathrm{H}^{j-2k-1}(V_{n,n-k-1}) \longrightarrow 0$$

$\mathrm{H}^{j-2k-1}(V_{n,n-k-1})$ is free abelian by induction. So this sequence splits. Hence, the middle term is free abelian. Odd dimensional classes $x = x_{2k+1} \in \mathrm{H}^{2k+1}(V_{n,n-k})$ maps to $1 \in \mathrm{H}^0(V_{n,n-k-1})$ under σ^*. Then $x^2 = -x^2 \implies 2x^2 = 0 \implies x^2 = 0$. Let

$$\Phi : \mathrm{H}^*(V_{n,n-k-1}) \otimes \Lambda(x) \to \mathrm{H}^*(V_{n,n-k}) \text{ be defined by}$$

$$\Phi(y \otimes x) = \pi^*(y) \smile x.$$

$$\Phi \text{ is algebra homomorphism since } x^2 = 0.$$

$$\text{Since } \sigma^*(\pi^*(y) \smile x) = (-1)^{\deg(y)} y \smile \sigma^*(x) = (-1)^{\deg(y)} y$$

$$\Phi \text{ is an additive isomorphism.}$$ □

Corollary 3.29. $H^*(U(n)) = \Lambda(x_1, x_3, \ldots, x_{2n-1})$

Example 3.3. *Let M^3 be the Euler class n circle bundle over S^2. Apply Gysin sequence.*

$$S^1 \longrightarrow M^3 \xrightarrow{\pi} S^2$$

$$0 = H^1(S^2) \longrightarrow H^1(M) \xrightarrow{\sigma^*} H^0(S^2) \xrightarrow{\smile\chi} H^2(S^2) \xrightarrow{\pi^*} H^2(M^3) \longrightarrow H^1(S^2) = 0$$

1. *If $\chi = 0$, then $H^*(M) = H^*(S^1 \times S^2)$.*

2. *If χ is a generator $[S^2] \in H^2(S^2)$, then $H^*(M) = H^*(S^3)$.*

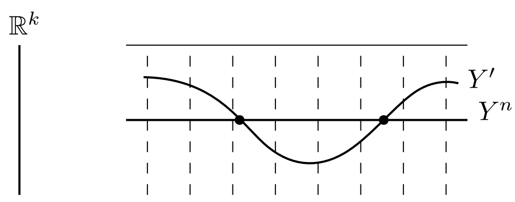
3. *If $\chi = k[S^2]$, then*

$$H^i(M) = \begin{cases} \mathbb{Z} \text{ for } i = 0, 3 \\ \mathbb{Z}_k \text{ for } i = 2 \ (k \neq 0) \\ 0 \text{ otherwise} \end{cases}$$

We can write $M^3 = D^2 \times S^1 \cup_f D^2 \times S^1 \longrightarrow S^2$, then by Van kampen

$$\pi_1 M = \langle x, y \mid x = y, 1 = y^k \rangle \cong \mathbb{Z}_k$$

Example 3.4. *We can compute the Euler class of an oriented $\mathbb{R}^k$- vector bundle over a closed smooth manifold Y, by pushing off a transverse copy of the zero section $Y' = i(Y)$ in a small disk bundle W of Y, and by taking the Poincare dual of the fundamental class of $[Y \cap Y'] \in H_*(Y)$. The Thom class of the zero section in W is $D[Y'] = D[Y] = \tau$*



$$\begin{tikzcd} D^k \arrow[r] & W^{n+k} \arrow[d, shift right] \\ & Y^n \arrow[u, bend right, "i"'] \end{tikzcd}$$

$$\begin{aligned} [Y \cap Y']_W &= [Y]_W \cdot [Y']_W = D^{-1}(DY' \smile DY) = (DY' \smile DY) \frown [W] \\ &= DY' \frown (DY \frown [W]) = \tau^k \frown [Y^n] \\ &= \left\langle \tau^k, i_*[Y] \right\rangle \text{ if } k = n \\ &= \langle i^*\tau, [Y] \rangle = \langle \chi, [Y] \rangle \end{aligned}$$

We used the identification $H^k(Y^k) \xrightarrow{\sim} H_0(Y)$, where $\chi \mapsto \langle \chi, [Y] \rangle$.

3.7 Leray Hirsch Theorem

Theorem 3.30 (Leray Hirsch)**.** *Let R be a commutative ring and $F \xrightarrow{i} E \xrightarrow{\pi} B$ a fiber bundle such that:*

(a) $H^*(F;R)$ is finitely generated free R-module for all $*$,

(b) There is $c_j \in H^{k_j}(E;R)$ such that $\{i^*(c_j)\}$ is a basis for $H^*(F;R)$ on each fiber F,

Then the following is an isomorphism:

$$\begin{array}{rccc} \Phi: & H^*(B;R) \otimes_R H^*(F;R) & \to & H^*(E;R) \\ & b \otimes i^*(c_j) & \mapsto & \pi^*(b) \smile c_j \end{array}$$

In other words, $H^*(E;R)$ is a free $H^*(B;R)$ module with basis $\{c_j\}$.

Proof. First, we consider the case B is finite dimensional CW complex. We prove by induction on dimension. Let $\dim(B) = n$, and $B' = B$ with a point deleted from the interior of each n-cell e^n_α. Set $E' = p^{-1}(B')$. Then we have a commutative diagram

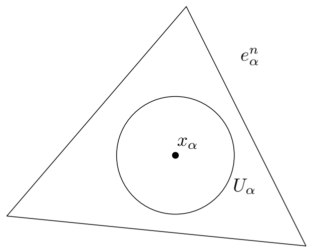
$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^*(B,B') \otimes_R \underline{\mathrm{H}^*(F)} \arrow[r] \arrow[d, "\Phi"', "\cong"] & \mathrm{H}^*(B) \otimes \underline{\mathrm{H}^*(F)} \arrow[r] \arrow[d, "\Phi"'] & \mathrm{H}^*(B') \otimes \underline{\mathrm{H}^*(F)} \arrow[r, "\delta"] \arrow[d, "\Phi"'] & \cdots \\ \cdots \arrow[r] & \mathrm{H}^*(E,E') \arrow[r] & \mathrm{H}^*(E) \arrow[r] & \mathrm{H}^*(E') \arrow[r, "\delta"] & \cdots \end{tikzcd}$$

free module

The map $\mathrm{H}^*(B,B') \otimes_R \mathrm{H}^*(F) \to \mathrm{H}^*(E,E')$ is defined as in the absolute case by using the relative cup product $\mathrm{H}^*(E,E') \otimes \mathrm{H}^*(E) \to \mathrm{H}^*(E,E')$. It is staightforward to check that the diagram commutes e.g.

$$\begin{tikzcd} \mathrm{H}^*(B') \otimes \mathrm{H}^*(F) & b \otimes i^*(c_j) \arrow[r, "\delta"] \arrow[d, "\Phi"] & \delta b \otimes i^*(c_j) \arrow[d] \\ & \pi^*(b) \smile c_j \arrow[d] & \pi^*(\delta b) \smile c_j \arrow[d, equal] \\ & \delta(\pi^* b \smile c_j) \arrow[r, equal] & \delta(\pi^*(b) \smile c_j) \end{tikzcd}$$

B' deformation retracts to B^{n-1}. $p^{-1}(B^{n-1}) \hookrightarrow E'$ is a weak homotopy equivalence. Hence inclusions on all cohomology groups.



$$U_\alpha \simeq \mathring{D}^n$$

$$\pi^{\ 1}(U_\alpha) \simeq U_\alpha \times F$$

By excision $\mathrm{H}^*(B,B') \cong \mathrm{H}^*(U,U')$ and $\mathrm{H}^*(E,E') \cong \mathrm{H}^*(\pi^{-1}(U),\pi^{-1}(U)')$, where $(U,U') \approx (D^n, S^{n-1})$. So it suffices to prove

$$\Phi : \mathrm{H}^*(U,U') \otimes_R \mathrm{H}^*(F) \to \mathrm{H}^*(U \times F, U' \times F)$$

is an isomorphism. Do this by relative Kunneth or do 5-lemma to diagram (B,B'). By induction the theorem holds for $U \simeq pt$ and $U' \simeq S^{n-1}$. $\square$

Example 3.5. *The normal bundle of the diagonal imbedding $d : N \cong \Delta \hookrightarrow N \times N$ is isomorphic to the tangent bundle of N. Now let $W = N^n \times N^n$, $\tau = \tau_\Delta^{N\times N} \in H^n(N\times N)$.*

$$\chi = d^*(\tau) = d^*(\tau|_\Delta) = d^*(\chi_\Delta^{N\times N}) \in H^n(N)$$

Theorem 3.31. *With field coefficients, let $\{\alpha\}$ be a basis of $H^*(N)$ and $\{\alpha^\circ\}$ be the dual basis, i.e. $\langle \alpha^\circ \smile \beta, [N]\rangle = \delta_{\alpha\beta}$ $\quad(\deg\alpha^\circ = n - \deg\alpha)$. Then we have*

$$\tau = \sum (-1)^{\deg\alpha} \alpha^\circ \times \alpha \in H^n(N \times N)$$

$$\textit{In particular,}\quad \chi = d^*\tau = \sum (-1)^{\deg\alpha}\alpha^\circ \smile \alpha$$

Proof. First write $\tau = \sum A_{\gamma\delta}\, \gamma^\circ \times \delta$ (by Kunneth, where $\deg\gamma = n - \deg\delta$) $\Longrightarrow$

$$\begin{aligned}
\langle(\alpha \times \beta^\circ) \smile \tau, [W]\rangle &= \langle \alpha \times \beta^\circ, \tau \frown [W]\rangle \\
&= \langle \alpha \times \beta^\circ, d_*[N]\rangle \\
&= \langle d^*(\alpha \times \beta^\circ), [N]\rangle \\
&= \langle \alpha \smile \beta^\circ, [N]\rangle \\
&= (-1)^{p(n-p)}\, \langle \beta^\circ \smile \alpha, [N]\rangle \\
&= (-1)^{p(n-p)} \delta_{\alpha\beta}
\end{aligned}$$

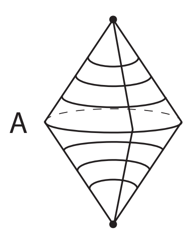
$$\begin{aligned}
\langle(\alpha \times \beta^\circ) \smile \tau, [W]\rangle &= \Big\langle (\alpha \times \beta^\circ) \smile \sum A_{\gamma\delta}(\gamma^\circ \times \delta), [W] \Big\rangle \\
&= (-1)^{p(n-p)} A_{\alpha\beta}\, \langle(\alpha \smile \alpha^\circ) \times (\beta^\circ \smile \beta), [N] \times [N]\rangle \\
&= (-1)^{(n-p)+p(n-p)} A_{\alpha\beta}\, \langle(\alpha^\circ \smile \alpha), [N]\rangle\, \langle(\beta^\circ \smile \beta), [N]\rangle \\
&= (-1)^{p(n-p)-p} A_{\alpha\beta}
\end{aligned}$$

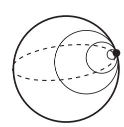
Therefore, $A_{\alpha\beta} = (-1)^p \delta_{\alpha\beta}$. $\square$

Chapter 4

Homotopy

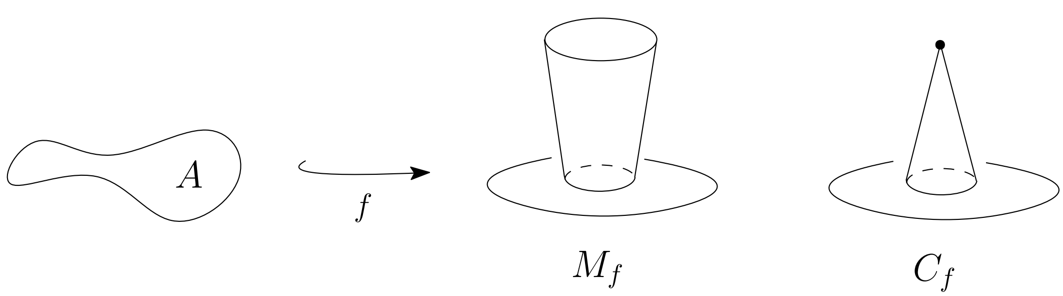
Suspension SA of A is obtained by crushing each boundary component of $A \times I$ to a single point, and reduced suspension of A is obtained from SA by crushing an arc to a single point; $\Sigma A = SA/a \times I \simeq SA$. ΣA has a natural base point, if (A, a) is a based space.





Reduced suspension

Set M_f = mapping cylinder, and C_f = mapping cone for $f : A \to X$.



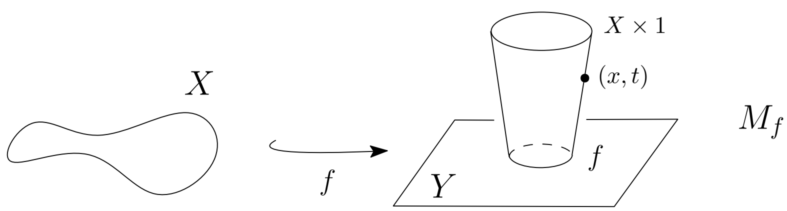
Define

$$X^I = \mathcal{P}X = \text{ Path space } = \{\sigma : I \to X \mid \sigma(0) = x_0\},$$
$$\Omega X = \text{ Loop space } = \{\sigma : I \to X \mid \sigma(0) = \sigma(1) = x_0\}.$$

Basic principals of homotopy theory in category of CW complexes (defined up to homotopy equivalences) and maps are:

(I) Every map $f : X \to Y$ is equivalent to inclusion $F : Y \to M_f$, where M_f is the mapping cylinder. That is, the following diagram homotopy commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "F"'] & Y \arrow[d, hook, "\simeq"] \\ & M_f \end{tikzcd}$$



$$M_f = Y \sqcup (X \times I)/(x,0) \sim f(x)$$

i is the inclusion $x \mapsto (x,1)$, and r a retraction such that $r \circ i = Id_Y$ and $i \circ r \simeq Id_X$

$$\begin{tikzcd} Y \arrow[r, hook, "i"'] & M_f \arrow[l, bend right, "r"'] \end{tikzcd}$$

r is a deformation retraction defined below, and $i \circ r \simeq Id_X$ is given by $(i \circ r)_s[x,t] = [x, st]$

$$\left\{ \begin{matrix} r|_Y = Id \\ r(x,t) = f(x) \end{matrix} \right\}$$

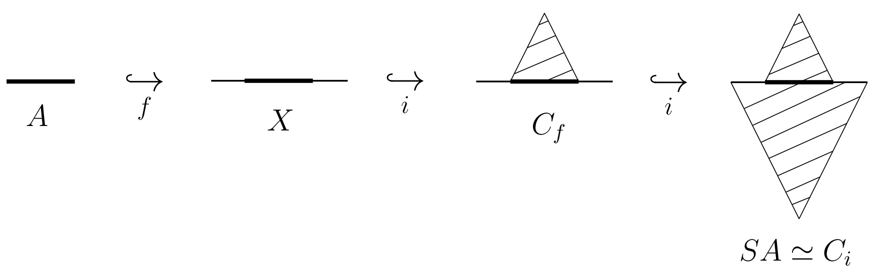
Definition 4.1. The sequence of maps $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_{n+1}$, is coexact if the following maps are exact, where we defined $f_n^{\#}(h) = h \circ f_n$.

$$[X_{n+1}, Y] \xrightarrow{f_n^{\#}} [X_n, Y] \xrightarrow{f_{n-1}^{\#}} \dots \xrightarrow{f_0^{\#}} [X_0, Y]$$

In each set the null homotopy class represents zero element, so exactness makes sense.

Example 4.1. *Every map $f : A \to X$ gives rise to so called Puppe coexact sequnce:*

$$\begin{tikzcd}
 & & & SA \arrow[r, "sf"] & SX & \\
 & & & C_i \arrow[u, phantom, "\wr"] & C_j \arrow[u, phantom, "\wr"] & \\
A \arrow[r, "f"] & X \arrow[r, "i"] & C_f \arrow[r, "j"] & C_i \arrow[r] & C_j \arrow[r] & C_k
\end{tikzcd}$$



This is equivalent to the following exact sequence (also holds with base points)

$$A \xrightarrow{f} X \to X/A \to SA \xrightarrow{Sf} SX \to SX/SA \to S^2A \to S^2X.$$

Example 4.2.

$$S^1 \to \mathbb{R}P^2 \to S^2 \to S^2 \to \mathbb{R}P^2$$

- Useful fact

$$\begin{array}{c} [\Sigma X, Y] = [X, \Omega Y] \\ \cup \\ f \longmapsto [x \mapsto f[x,t]] \end{array}$$

$$f[x_0, t] = y_0 \quad \Longrightarrow \quad x_0 \mapsto f[x_0, t]$$

Definition 4.2. Given $p : E \to B$ and $X \xrightarrow{f} B$ we say $\widetilde{f} : X \to E$ is a lifting of f if $p \circ \widetilde{f} = f$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\widetilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

Definition 4.3. We say $p : E \to B$ satisfies homotopy lifting property (HLP) with respect to (X, A) if every homotopy $f_t : X \to B$ lifts to a homotopy $\widetilde{f_t} : X \to E$ starting with a given lifting $\widetilde{f_0} : X \to E$ (and extending a given lifting $\widetilde{f_t} : A \to E$).

$$\begin{tikzcd} (X \times 0) \cup (A \times I) \arrow[r] \arrow[d, hook] & E \arrow[d] \\ X \times I \arrow[ur, "\widetilde{F}"] \arrow[r, "F"] & B \end{tikzcd}$$

Since $(D^n \times I, D^n \times \{0\}) \simeq (D^n \times I, D^n \times \{0\} \smile (S^{n-1} \times I))$, satisfying HEP with respect to D^n is equivalent to satisfying HEP with respect to (D^n, S^{n-1}).

Definition 4.4. A map $p : E \to B$ satisfying HLP with respect to D^k is called (Serre) fibration. Then, it also satisfies HLP with respect to all CW complexes (use Theorem 1.4).

- Locally trivial fiber bundle $F \to E \underset{p}{\to} B$ is a fibration.
- Pullback fibration

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

$$\text{is defined by} \quad f^*E = \{(x, e) \in A \times E \mid f(x) = p(e)\}\,.$$

Proposition 4.5. *If $f_0 \simeq f_1$ then f_0^*E and f_1^*E are fiber homotopy equivalent.*

Proof. Let $F : A \times I \to B$ be homotopy we get $F^*E \to A \times I$.

$$F^*E = \{(a,t,e) \mid F(a,t) = p(e)\}$$

Then, basically by applying the unique path lifting (HLP) property to the paths $t \mapsto (a,t)$, we get a fiber homotopy equivalence $f_0^*E = F^*E|_{A\times 0} \ \simeq \ F^*E|_{A\times 1} = f_1^*E$. $\square$

Every map is equivalent to a fibration:

Up to homotopy, we can turn any map $f : A \to B$ into a fibration. This means that after replacing A by a homotopy equivalent copy, we can make f a fibration. Let B^I be the space continous maps $I = [0,1] \to B$ with compact open topology, and define

$$E_f = \left\{(a,\gamma) \in A \times B^I \mid f(a) = \gamma(0)\right\}.$$

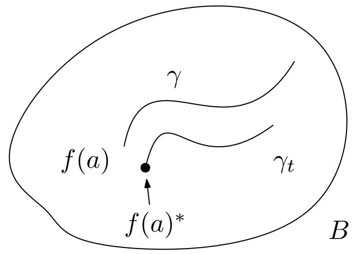
For $b \in B$, let b^* be the constant path $I \to B$ mapping to b. Then we can define maps

$$\begin{tikzcd} X \arrow[r, "i"] & E_f \arrow[l, bend left, "j"] \end{tikzcd}$$

$i(a) = (a, f(a)^*)$ and $j(a,\alpha) = a$ that are homotopy inverses of each other.

$$\begin{tikzcd} E_f \arrow[d, bend left, "j"] \arrow[dr, dashed, "\pi"] & \\ A \arrow[u, bend left, "i"] \arrow[r, "f"'] & B \end{tikzcd} \qquad \pi(a,\gamma) = \gamma(1)$$

Clearly $j \circ i(a) = a$, and $i \circ j(a,\gamma) = (a, f(a)^*)$. We construct a homotopy $F_t; E_f \to E_f$ giving $i \circ j \simeq Id_{E_f}$ as follows: let γ_t be the portion of the path γ with initial point $\gamma(0)$ and end point $\gamma(t)$. Then $F_t(a,\gamma) = (a, \gamma_t)$ gives the required homotopy.



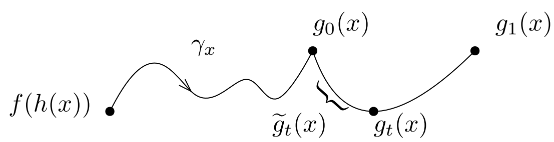
Proposition 4.6. *$E_f \xrightarrow{\pi} B$ is a fibration where $\pi(a,\gamma) = \gamma(1)$. Note $A \simeq E_f$.*

Proof. Let us check HLP property. Given the following diagram with initial lifting $\widetilde{g}_0 : X \to E_f$ of $g_0 : X \to B$, we need to find a lifting $\widetilde{g}_t : X \to E_f$ of $g_t : X \to B$.

$$\begin{tikzcd}
 & E_f = \left\{(a,\gamma) \in A \times B^I \mid f(a) = \gamma(0)\right\} \arrow[d, "\pi"'] \arrow[d, dashed, shift left=8] & \\
X \arrow[ur, "\widetilde{g}_0"] \arrow[r, "g_t"'] & B \ni \gamma(1) & \pi \circ \widetilde{g}_1 = g_1
\end{tikzcd}$$

So we have $\widetilde{g}_0(x) = (h(x), \gamma_x)$, $f(h(x)) = \gamma_x(0)$ with $\pi\widetilde{g}_0(x) = \gamma_x(1) = g_0(x)$. For each $x \in X$, let $\widetilde{\gamma}_x(t) = \gamma_x(t) \cdot \gamma_t(x)$ be the concatenation of the paths in B, then define

$$\widetilde{g}_t(x) = (h(x), \gamma_x \cdot g_t(x))$$



$$\begin{tikzcd}
 & \widetilde{g}_t(x) = (h(x), \gamma_x \cdot \widetilde{g}_t(x)) \arrow[d] \\
x \arrow[ur] \arrow[r, dashed] & \gamma_x \cdot \widetilde{g}_t(x)(1)
\end{tikzcd}$$

$\square$

Next we analyze what happens if the original map f is already a fibration $\pi : E \to B$.

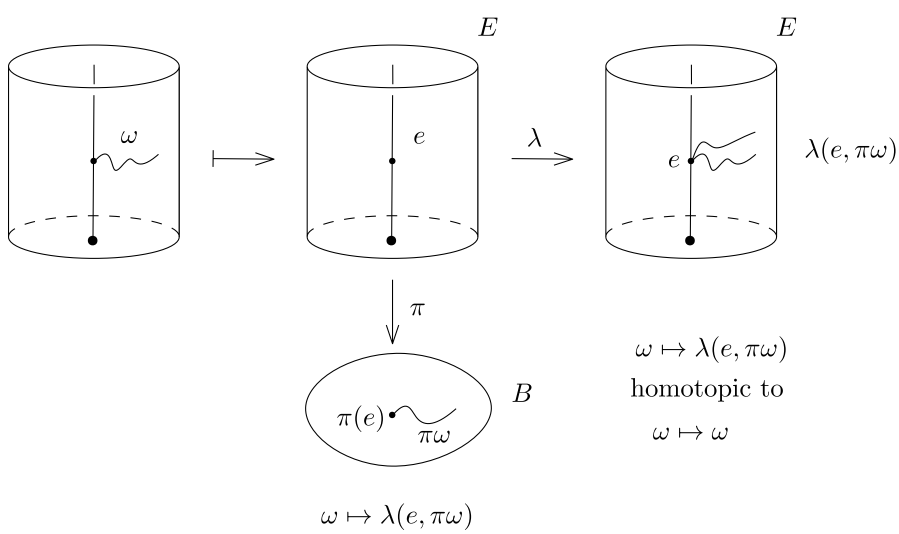
$$\begin{tikzcd}
F = \pi^{-1}(b_0) \arrow[r] & E \arrow[r, "\pi"] \arrow[d, hook] & B \\
& E_p = E' = \left\{(e,\beta) \in B^I \mid \pi(e) = \beta(0)\right\} \arrow[ur] &
\end{tikzcd}$$

Let us first mention the fibration version of unique homotopy lifting.

Lemma 4.7. *Let $\lambda : E' \longrightarrow E^I$ be "the lifting of a path starting at e" map i.e. $\lambda(e,\beta)(0) = e$, and $\pi\lambda(e,\beta) = \beta$. Then the map $E^I \longrightarrow E' \xrightarrow{\lambda} E^I$ given by*

$$\omega \mapsto (\omega(0), \pi\omega) \mapsto \lambda(\omega(0), \pi\omega)$$

is homotopic to Id. The proof follows HLP property of fibrations.

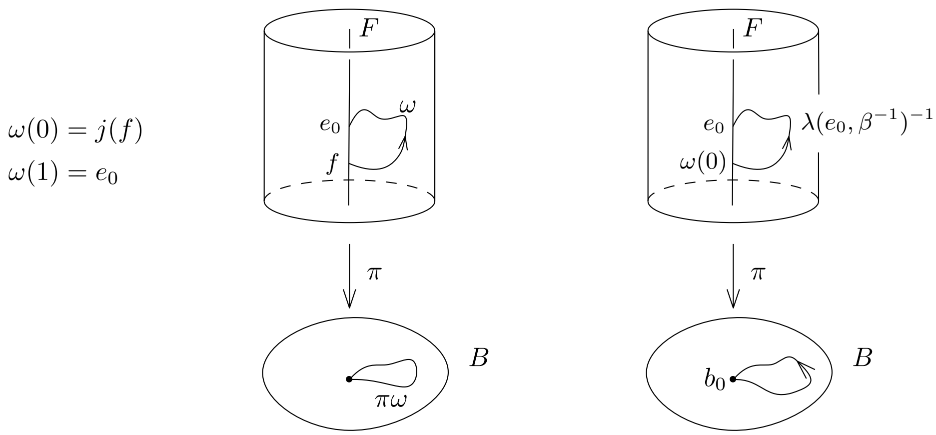


Lemma 4.8. *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration, j be the inclusion of a fiber $\pi^{-1}(b_0)$. Let $F' = E_j \xrightarrow{j'} E$ be the fibration obtained from the inclusion j, by the process of turning j into fibration (Proposition 4.6), and $G = (j')^{-1}(e_0)$ be the fiber of j'. Then $G \simeq \Omega B$.*

$$\begin{tikzcd}
\Omega B & & F \arrow[r, "j"] \arrow[d, hook] & E \arrow[r, "\pi"] & B \\
G = (j')^{-1}(e_0) \arrow[u, "\simeq"'] \arrow[rr, hook] & & F' \arrow[ur, "j'"'] \arrow[d, equal] & & j'(f,\omega) = \omega(1) \\
\Omega B \ \simeq \ G \arrow[rr, hook] & & \big\{(f,\omega) \in F \times E^I \mid f = \omega(0)\big\} & &
\end{tikzcd}$$

Proof. We will define $\varphi : G \to \Omega B$, $\psi : \Omega B \to G$ such that $\varphi \circ \psi \simeq Id$ and $\psi \circ \varphi \simeq Id$

$$G = (j')^{-1}(e_0) = \big\{(f,\omega) \in F \times E^I \mid f = \omega(0),\ e_0 = \omega(1)\big\}$$



$$\begin{tikzcd}
G \arrow[r, "\varphi"] & \Omega B \arrow[l, bend left, "\psi"]
\end{tikzcd}$$

$$\begin{aligned}
\varphi(f,\omega) &= \pi\omega \\
\psi(\beta) &= (\lambda(e_0, \beta^{-1})^{-1}(0), \lambda(e_0, \beta^{-1})^{-1})
\end{aligned}$$

$$\begin{aligned}
\varphi\psi(\beta) &= \pi\lambda(e_0, \beta) = \beta \\
\psi\varphi(f,\omega) &= (\lambda(e_0, \pi\omega^{-1})^{-1}(0),\ \lambda(e_0, \pi\omega^{-1})^{-1} \simeq (f,\omega)
\end{aligned}$$

The last equivalence follows from $\lambda(\omega(0), \pi\omega) \simeq \omega$ (Lemma 4.7). ☐

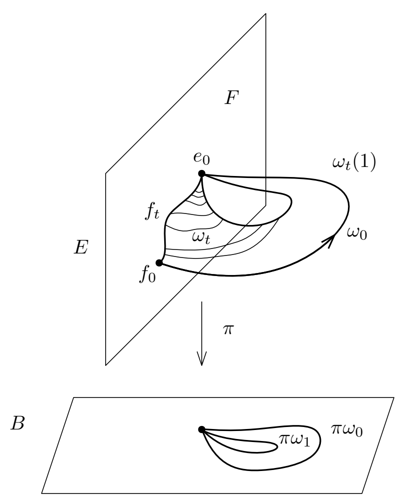
We have constructed a fibration $\Omega B \simeq G \to F' \to E$, we can now iterate this process by making the fiber inclusion $k : G \to F'$ a fibration, and get another fibration $k' : G' \to F'$, and then take its fiber $H = (k')^{-1}(e_0, e_0^*)$:

$$\begin{tikzcd}
\Omega E \arrow[r, "\Omega\pi"] & \Omega B & F \arrow[r, "j"] & E \arrow[r, "\pi"] & B \\
\Omega E \arrow[u, "\simeq", "T"'] & G \arrow[u, "\simeq", "\gamma"'] \arrow[r, "k"] & F' \arrow[u, "\simeq"] \arrow[ur, "j'"] & & \\
H \arrow[u, "\simeq", "\alpha"'] \arrow[r, "l"] & G' \arrow[u, "\simeq", "\beta"'] \arrow[ur, "k'"] & & &
\end{tikzcd} \tag{4.1}$$

$$\begin{aligned}
F' &= \big\{(f,\omega) \in F \times E^I \mid f = \omega(0)\big\} \quad \text{and} \quad j'(f,\omega) = \omega(1) \\
G &= (j')^{-1}(e_0) = \big\{(f,\omega) \in F \times E^I \mid f = \omega(0),\ e_0 = \omega(1)\big\} \\
G' &= \{(f,\omega,f_t,\omega_t) \in F \times E^I \times F \times E^I \mid (f_0,\omega_0) = (f,\omega), \\
&\qquad f = \omega(0),\ e_0 = \omega_0(1),\ f_t = \omega_t(0)\} \\
&= \big\{(f_t,\omega_t) \in F \times E^I \mid f_t = \omega_t(0), e_0 = \omega_0(1)\big\}
\end{aligned}$$

$$k'(f_t,\omega_t) = (f_1,\omega_1) \text{ and let } H = (k')^{-1}(e_0,e_0^*), \text{ then}$$

$$H = \{(f_t,\omega_t) \mid f_t = \omega_t(0),\ e_0 = \omega_0(1),\ (f_1,\omega_1) = (e_0,e_0^*)\}$$



The identification $H \simeq \Omega E$ is given by $(f_t, \omega_t) \mapsto \omega_t(1)$. By inserting the obvious maps α, β, γ, T; we get a commuting the diagram (4.1) (T is parametrization reversing map).

$$\gamma\beta(f_t, \omega_t) = \gamma(f_0, \omega_0) = \pi\omega_0 = \Omega\pi(\omega_0)$$
$$(\Omega\pi)\alpha(f_t, \omega_t) = (\Omega\pi)(\omega_t(1)) = T(\Omega\pi)\omega_t(1)$$

By iterating this process, we can extend any given fibration $F \xrightarrow{j} E \xrightarrow{\pi} B$ to fibrations

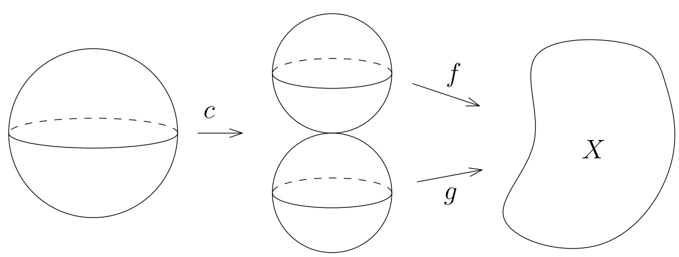
$$\cdots \to \Omega F \xrightarrow{\Omega j} \Omega E \xrightarrow{\Omega\pi} \Omega B \longrightarrow F \xrightarrow{j} E \xrightarrow{\pi} B.$$

4.1 Homotopy Groups

$$I^n = I \times \cdots \times I$$

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$$

We denote representatives of $f : (I^n, \partial I^n) \to (X, x_0)$ by $[f] \in \pi_n(X, x_0)$. Group structure



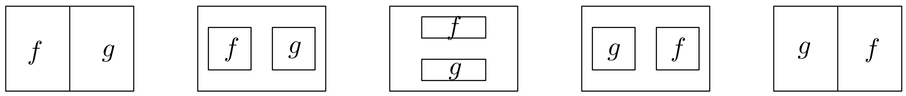
$$(f+g)(s_1, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & \text{if } s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \ldots, s_n) & \text{if } s_1 \in [\frac{1}{2}, 1] \end{cases}$$

We can also identify $\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$. Any map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism, which is functorial with respect to maps.

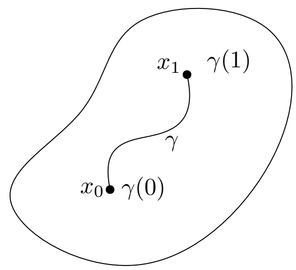
$$f_\# : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

- $\pi_1(X, x_0) =$ fundamental group

- $\pi_n(X, x_0)$ is abelian when $n \geq 2$. The following pictures explains this.



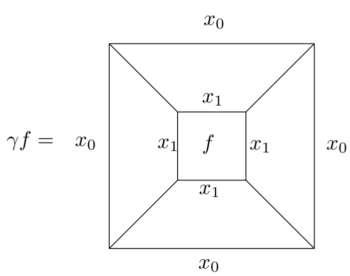
- If X is path connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$.



More specifically any path γ connecting x_0 to x_1 defines an isomorphism

$$\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$$

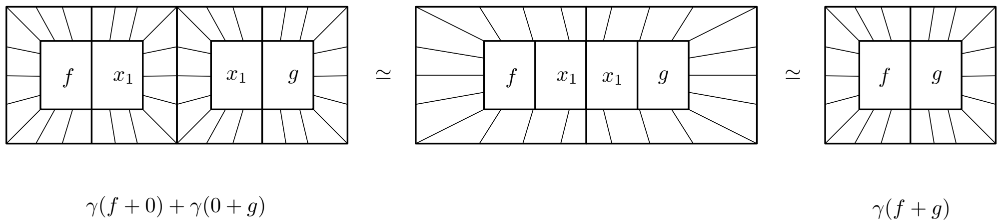
by $\beta_\gamma([f]) = [\gamma f]$. Note that $\beta_\gamma^{-1} = \beta_{\gamma^{-1}}$.



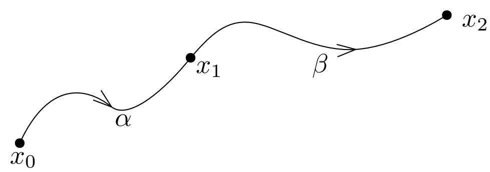
$$\left\{ \begin{array}{c} \gamma \simeq \gamma' \\ f \simeq f' \end{array} \right\} \implies \gamma f \simeq \gamma' f'$$

$$\begin{aligned} \gamma(f+g) &\overset{h_t}{\simeq} \gamma f + \gamma g \\ (\gamma f) g &\simeq g(\gamma f) \\ 1f &\simeq f \end{aligned}$$

$$\text{where } \; h_t(s_1, \dots, s_n) = \begin{cases} \gamma(f+0)((2-t)s_1, s_2, \dots, s_n) \\ \gamma(0+g)((2-t)s_1 + t - 1, s_2, \dots, s_n) \end{cases}$$



$$\beta_{\alpha\beta} = \beta_\alpha \circ \beta_\beta$$



Pick $\gamma \in \pi_1(X, x_0)$ then $[f] \mapsto \beta_\gamma$ defines an action of π_1 on π_n, satisfying $\beta_{\alpha\beta} = \beta_\alpha \circ \beta_\beta$

$$\pi_1(X, x_0) \to \mathrm{Aut}(\pi_n(X, x_0))$$

- When $n = 1$ this action is inner automorphism

$$[\gamma] \mapsto ([f] \mapsto [\gamma^{-1}][f][\gamma])$$

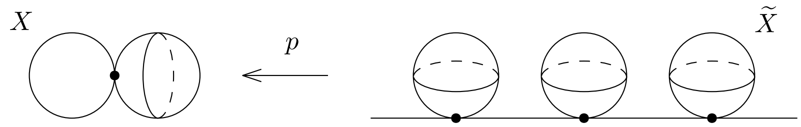
- When $n \geq 2$, $\pi_n(X, x_0)$ is an $\mathbb{Z}[\pi_1(X, x_0)]$-module

$$\mathbb{Z}[\pi_1(X, x_0)] = \left\{ \sum n_i \gamma_i \mid \gamma_i \in \pi_1(X, x_0) \right\}$$
$$\left(\sum n_i \gamma_i \right) \alpha = \sum n_i (\gamma_i \alpha)$$

- $X \subset Y \implies \pi_n(X) \to \pi_n(Y) \to \pi_n(Y, X) \xrightarrow{\partial} \pi_{n-1}(X)$
- $F \to E \to B$ is a fibration, then $\pi_1(B)$ acts on $H_*(F)$.

Definition 4.9. X is *n-simple* if the action of π_1 on π_n is trivial.
We say X is *simple* it it is n-simple for all n.

Example 4.3. *L* $X = S^1 \vee S^n$ *and let* $\widetilde{X}$ *be its universal cover*



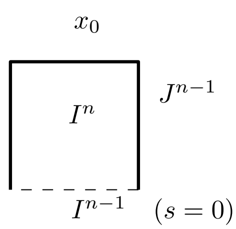
$$p_* : \pi_n(\widetilde{X}, \widetilde{x}_0) \to \pi_n(X, x_0) \qquad n \geq 2$$

Therefore as a $\mathbb{Z}[t, t^{-1}]$*-module* $\pi_n(X) \cong \pi_n(\widetilde{X}, \widetilde{x}_0) \cong \mathbb{Z}[t, t^{-1}]$.

4.1.1 Relative homotopy groups

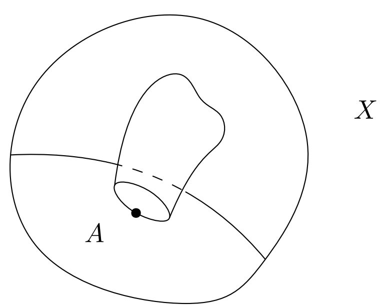
Given $A \subset X$. Write $I^n = I^{n-1} \times I$, and let $J^{n-1} = \partial I^n - I^{n-1} \times \{0\}$, then define

$$\pi_n(X, A, x_0) = \left[\left(I^n, \partial I^n, J^{n-1}\right), (X, A, x_0)\right]$$



$A \subset X$

$\pi_n(X, A, x_0) =$



- $\pi_n(X, A, x_0)$ is abelian for $n \geq 3$,
- $\pi_1(X, A, x_0)$ is not a group and π_0 is not defined.

Theorem 4.10. *If* $x_0 \in A \overset{i}{\hookrightarrow} X$*, then there is the following long exact sequence:*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \dots$$

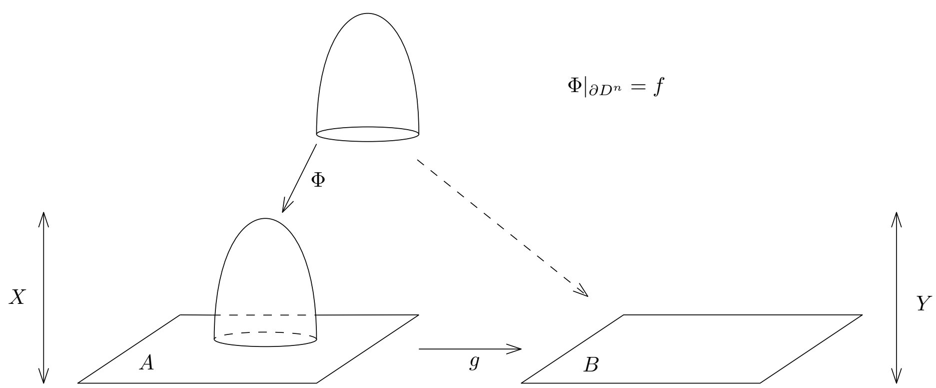
Proposition 4.11 (Compression Lemma)**.** *If (X, A) a CW pair with $X = A \cup_f e^n$ and $g : (X, A) \to (Y, B)$ be a map, and $\pi_n(Y, B, y_0) = 0$, then $g \simeq g'$ rel A such that $g'(X) \subset B$.*

Proof. Consider the following compostion

$$\begin{tikzcd}
(D^n, S^{n-1}) \arrow[r, "\Phi"] \arrow[rr, bend right, "g\circ\Phi\simeq h"'] & (X, A) \arrow[r, "g"] & (Y, B)
\end{tikzcd}$$

where Φ is the characteristic map of f. $g \circ \Phi \simeq h$ rel S^{n-1} with $h(D^n) \subset B$. Let

$$g'(x) = \begin{cases} g(x) & \text{on } A \\ h(x) & \text{on } D^n \end{cases}$$

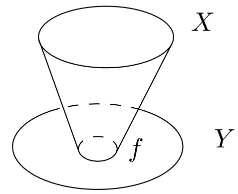


By using the homotopy extension property, we can repeat this process for other cells. $\square$

Theorem 4.12 (Whitehead)**.** *If a map $f : X \to Y$ between CW-complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ $\forall\, n$, then it is a homotopy equivalence.*

Proof. After replacing Y by its homotopy equivalent copy M_f, we can assume f is an inclusion $i : X \hookrightarrow Y$. There is the obvious deformation retraction $r : M_f \to Y$.

$$\begin{tikzcd} & M_f \arrow[dl, dashed, bend right, "s"'] \arrow[d, shift right, "r"] \\ X \arrow[ur, hook, "i"] \arrow[r, "f"] & Y \arrow[u, hook, shift right, "j"] \end{tikzcd}$$

$Y \simeq M_f =$ 

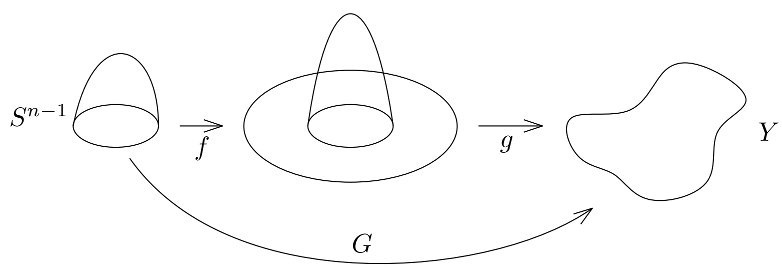
Also since $\pi_i(M_f, X) = 0 \ \forall i$, applying Proposition 4.11 to $Id : (M_f, X) \to (M_f, X)$ gives a deformation retraction of $s : M_f \to X$ with $i \circ s \simeq Id$ rel X. Then

$$\begin{aligned} f \circ (s \circ j) = (r \circ i) \circ (s \circ j) &= r \circ (i \circ s) \circ j = r \circ j \simeq Id_Y, \\ (s \circ j) \circ f = (s \circ j) \circ (r \circ i) &= s \circ (j \circ r) \circ i \simeq s \circ i \simeq Id_X. \end{aligned}$$

□

4.1.2 Cellular Approximation

If (X, A) is a CW pair with $X = A \bigcup \cup e^n$, and Y is path connected with $\pi_{n-1}(Y) = 0$, then every $g : A \to Y$ has an extension $G : X \to Y$.

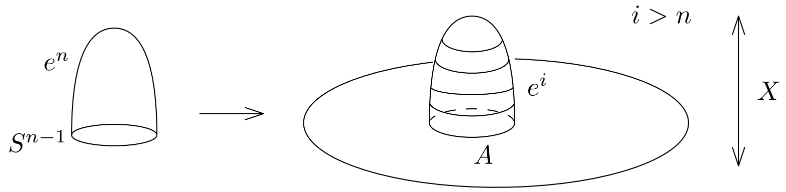


Definition 4.13. Let X and Y be CW complexes. $f : X \to Y$ is cellular if $f(X^n) \subset Y^n$ for all n.

Next we need to recall cellular approximation Theorem 1.6.

Theorem 4.14. *Any map $f : X \to Y$ between CW complexes is homotopic to a cellular map. If $A \subset X$ is a subcomplex and $f|_A$ cellular, we can assume the homotopy fixes A.*

A CW pair (X, A) is n-connected if $X = A \cup \Sigma e^i \ i > n$. In particular, (X, X^n) is n-connected implies $X^n \hookrightarrow X$ induces isom for π_i for $i < n$.



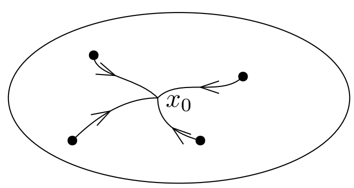
$\pi_i(X, A)$

Proposition 4.15. *If $\pi_n(X) = 0 \ \ \forall \ n < N$, then there is homotopy $f_t : X \to X$ with*

$$\begin{Bmatrix} f_0 = Id \\ f_1(X^{(N-1)}) = x_0 \end{Bmatrix}$$

$$\textit{In particular } \widetilde{H}_k(X) = 0 \qquad \forall k < N$$

Proof of Proposition. Since $\pi_0(X) = 0$, there is homotopy $f_t : X^{(0)} \to X$ with $f_0 = id$, $f_1(X^{(0)}) = \{x_0\}$. Use HEP to extend $f_t : X \to X$ with $f_0 = Id$, $f_1(X^{(0)}) = \{x_0\}$.



Now consider $X^{(1)}$, then $f_1(e_1)$ = loop based at x_0. Use $\pi_1(X, x_0) = 0$ to get a homotopy rel $X^{(0)}$ to construct map $f_2 : X^{(1)} \to x_0$. By homotopy extension, we can find homotopy $f_t : X \to X$ for $1 \leq t \leq 2$ with $f_2|_{X^{(1)}} = x_0$. Continue this fashion by using the fact $\pi_n(X) = 0$ for $n < N$, get $f_N : X \to X$ with $f_N(X^{(N-1)}) = x_0$. $\square$

Definition 4.16 (Hurewicz map)**.** Let $[S^n]$ represent the fundamental class of S^n

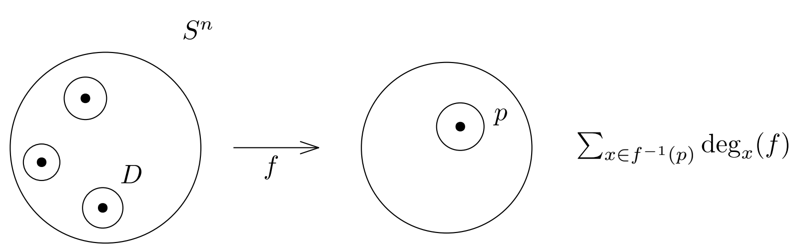
$$h : \pi_n(X, x_0) \to H_n(X)$$
$$[f] \mapsto f_*[S^n]$$

Theorem 4.17 (Brouwer)**.** *The Hurewicz map is an isomorphism.*

$$h : \pi_n(S^n, s_0) \xrightarrow{\cong} H_n(S^n)$$

Proof. Pick a regular value $p \in S^n$ of $f : S^n \to S^n$, where $[f] \in \pi_n(S^n) = \mathbb{Z}$ is a generator. Then $f^{-1}(p) = \{p_i\}$ consist of finitely many points. Hence, we can pick disjoint balls $\{D_i\}$ centered at $\{p_i\}$, such that f maps each D_i diffeomorphically to a fixed small ball D centered at p. If $f_i := f|_{D_i} : D_i \to D$ is orientation preserving (or reversing), assign p_i the number $+1$ (or -1), and call this number local degree of f, and denote by $deg(p_i)$. By composing f with the collapse homotopy equivalence $S^n \to S^n/(S^n - D) \simeq S^n$, we can assume f maps $C := S^n - \cup D_i$ to a point. Then its clear that Hurewicz map is induced by first collapsing $S^n \to S^n/C$, then mapping resulting bouquet of spheres $\vee S^n_i$ to S^n, where the map on each sphere is obtained by f_i (here we are using CW-homology).

$$h[f] = \sum_{p_i \in f^{-1}(p)} deg(p_i)[S^n] = deg(f)[S^n]$$



□

Theorem 4.18 (Hurewicz)**.** *Let X be a CW complex with $\pi_k(X) = 0$ for all $k < n$ then*

(1) $\widetilde{H}_k(X) = 0$ *for all* $k < n$

(2) $h : \pi_n(X) \xrightarrow{\cong} H_n(X)$

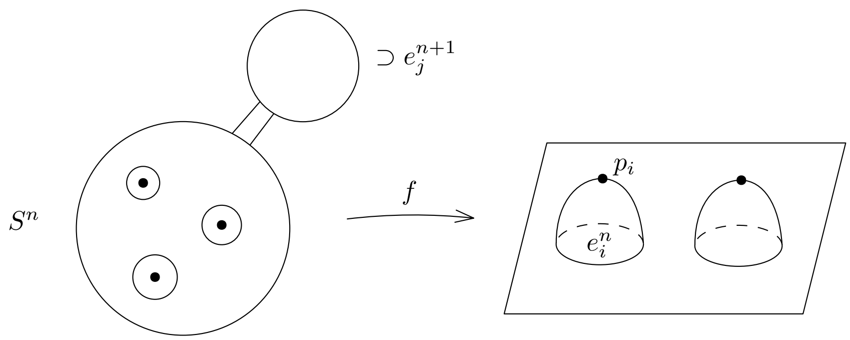
Proof. (1) Follows from Proposition 4.15. Next let us prove h is onto. Since $\pi_i(X) = 0$ for $i < n$, by Proposition 4.15 there is $f : X \to X$ such that $f \simeq Id$ and $f(X^{(n-1)}) = x_0$.

Let $a \in \mathrm{H}_n(X)$, by using CW-homology write $a = \sum a_\alpha e^n_\alpha$, consider

$$f|_{e^n_\alpha} : (e^n_\alpha, \partial e^n_\alpha) \to (X, x_0)$$

Hence $[f|_{e^n_\alpha}] \in \pi_n(X, x_0)$. Therefore $a = h(\sum_\alpha \left[f|_{e^n_\alpha}\right])$. Also h is a homomorphism.

Next we prove $h : \pi_n(X) \longrightarrow \mathrm{H}_n(X)$ is $1-1$. Let $f : S^n \to X = X^{(n-1)} \cup \sum e^n_i$ with $f_*[S^n] = 0$ in $\mathrm{H}_n(X)$. Make f transversal to the tips $\{p_i\}$ of the n cells e^n_i, $f \pitchfork p_i$.



Let $\lambda_i = \sum_{x \in f^{-1}(p_i)} \deg(f, x)$. Then the chain $c = \sum \lambda_i [e^n_i] = h(f)$ is a cycle in $C_n(X)$ representing $f_*[S^n]$. Since $f_*[S^n] = 0$, $c = \partial c'$ i.e. $\sum \lambda_i [e^n_i] = \partial \left(\sum \mu_j e^{n+1}_j\right) = \sum \mu_j (\partial e^{n+1}_j)$. Adding the map $f : S^n \to X^{(n)}$ the linear combination. $\sum \mu_j \partial e^{n+1}_j$ makes each $\lambda_i = 0$. Clearly this does not change the homotopy class $f : S^n \to X$.

Recall again, by Proposition 4.15 there is a map $\psi : X \to X$ with

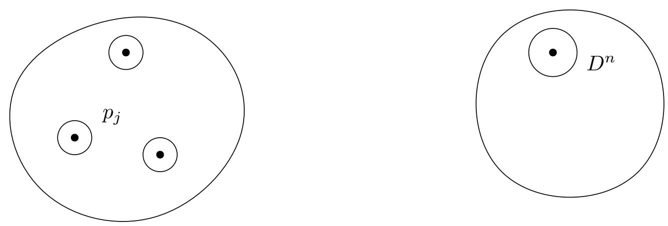
$$\begin{Bmatrix} \psi \simeq Id_X \\ \psi|_{X^{(n-1)}} = const \end{Bmatrix}$$

$$\begin{tikzcd}
S^n \arrow[rr, "f", "\text{each } \lambda_i = 0"'] \arrow[drrr, bend right] & & X^{(n)} \arrow[rr, "\psi|_{X^{(n)}}"] \arrow[dr, "p"] & & X \\
& & & \vee S^n \arrow[ur, "g"'] &
\end{tikzcd}$$

$p \circ f$ homologous to zero implies $p \circ f \simeq 0$. Thus $f \simeq 0$, and so [f]=0 in $\pi_n(X)$ $\square$

Proposition 4.19. $h : \pi_n\left(\vee_{i=1}^r S_i^n\right) \underset{\cong}{\longrightarrow} H_n\left(\vee_{i=1}^r S_i^n\right) = \oplus_{i=1}^r \mathbb{Z}$

Proof. Pick $f : S^n \to \vee S^n$ with $p_i \in S_i^n \setminus \{*\}$, such that $f \pitchfork p_i$, and let $\deg_{p_i}(f) = \sum \pm f^{-1}(p_i)$. If $\deg_{p_i}(f) = 0$, $f \simeq$ map missing p_i. First check $r = 1$ case.



This means that if $\pi_i : \vee_{i=1}^r S_i^n \to S_i^n$ is the obvious map, then $h[f]$ is given by

$$[f] \mapsto (deg_{p_1}(f \circ \pi_1), .., deg_{p_r}(f \circ \pi_r))$$

By the argument of Theorem 4.17 this is an isomorphism. □

Theorem 4.20 (Relative Hurewicz)**.** *Assume $A \subset X$, such that $\pi_0 A = \pi_1 A = 0$, and also $\pi_i(X, A) = 0 \quad \forall\, i < n$ $(n \geq 2)$ then:*

$$H_i(X, A) = 0 \quad i < n$$

$$\pi_n(X, A) \underset{\cong}{\xrightarrow{h}} H_n(X, A) \cong H_n(X/A).$$

Proof. By Proposition 4.15 there is homotopy $\varphi_t : X \to X$ such that

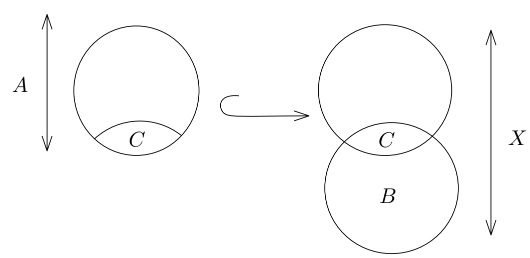
1. $\varphi_0 = Id$
2. $\varphi_t|_A = Id$
3. $\varphi(X^{n-1}) \subset A$,

This implies $\mathrm{H}_i(X, A) = 0$ for $i < n$ (recall chains in CW-homology).

$$\begin{tikzcd} \pi_n(X, A) \arrow[r, "h"] & H_n(X, A) \arrow[r, "\varphi_1", "\cong"'] & H_n(X, A) \arrow[ll, bend left, "g"] \end{tikzcd}$$

$\varphi_1 \circ h$ is obviously onto. To prove surjectivity of h we define its inverse g as follows: Define $g\left(\sum n_i e_i^n\right) = \sum n_i[\sigma_i]$ where $\sigma_i : (D^n, \partial D^n) \to (X, A)$ describing e_i^n. Now check that g is well defined, i.e. $g(\partial\sigma_{n+1}) = 0$. $\square$

Theorem 4.21 (Homotopy Excision)**.** *Given CW complexes $X = A \cup B$, $C = A \cap B$.*



If (A, C) is m-connected and (B, C) is n-connected then the induced map

$$\begin{aligned} \pi_i(A, C) &\xrightarrow{\cong} \pi_i(X, B) \qquad \forall i < m + n \\ \pi_{m+n}(A, C) &\twoheadrightarrow \pi_{m+n}(X, B) \qquad \textit{onto} \end{aligned}$$

Corollary 4.22 (Freudenthal suspension theorem)**.**

$$\pi_i(S^n) \underset{\cong}{\overset{S}{\to}} \pi_{i+1}(S^{n+1}) \qquad \forall i < 2n - 1$$
$$\pi_{2n-1}(S^n) \twoheadrightarrow \pi_{2n}(S^{n+1})$$

More generally, the same holds for X, when X is $(n-1)$-connected CW complex.

$$\pi_i(X) \xrightarrow{S} \pi_{i+1}(SX) \ \textit{is} \ \begin{cases} \cong & \textit{for } i < 2n - 1 \\ \textit{onto} & \textit{for } i = 2n - 1 \end{cases}$$

Proof. Note that $SX = C_+X \cup C_-X$ and $C_-X \cap C_+X = X$. First check

$$\pi_i(C_\pm X, X) \cong \pi_{i-1}(X) = 0 \qquad \forall\, i - 1 \leq n - 1$$

then apply Theorem 4.21 to the pair (C_+X, X) and (C_-X, X) and get

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \stackrel{\cong}{\to} \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX) \quad \forall i < 2n-1 \qquad \square$$

Proof of excision. (sketch) We prove this by induction on cells. Consier the case:

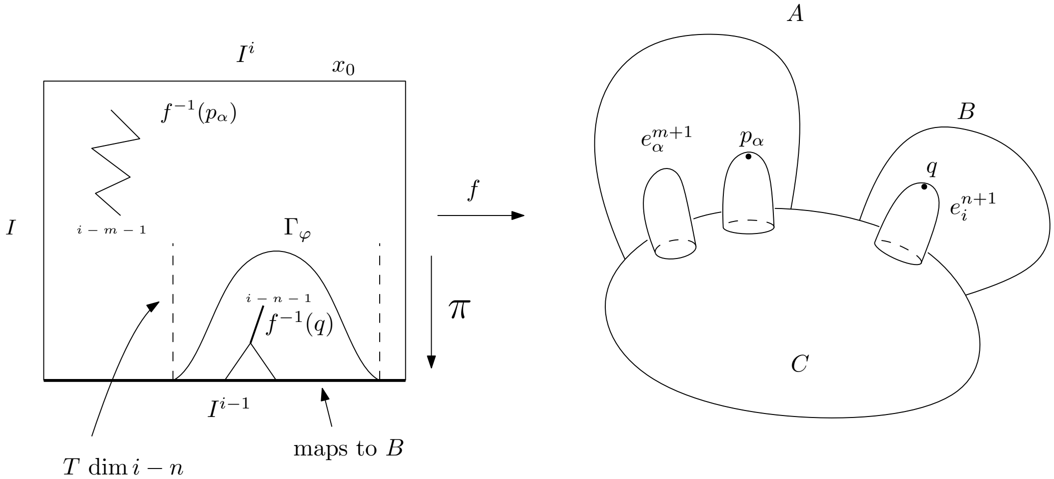
$$A = C \cup \sum e_\alpha^{m+1} \text{ and } B = C \cup e^{n+1}$$

Let us indicate how to prove $\pi_i(A, C) \twoheadrightarrow \pi_i(X, B)$ is onto for $i < m+n$. Now pick $f : (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$. If $\mathrm{im}(f)$ compact then it meets finitely many cells. Let Let $\{p_\alpha\}$ be the tips of the cells $\{e_\alpha^{m+1}\}$, and q be the tip of the cell e^{n+1}. Make f transverse to these points.Then we have dim $f^{-1}(p_\alpha) = i-m-1$ dim $f^{-1}(q) = i-n-1$.

Now let $\pi : I^i = I \times I^{i-1} \to I^{i-1}$ be the projection (to the bottom floor), and let $T = \pi^{-1}(\pi(f^{-1}(q)))$ be the points that lie over $\pi(f^{-1}(q))$. Clearly $dim(T) = i - n$, therefore after a small isotopy the sets T and $\cup f^{-1}(e_\alpha)$ don't meet since by hypothesis

$$(i-n) + (i-m-1) - i < 0$$

So we can can push $f^{-1}(q)$ outside of I^i by a straight down homotopy away from $\cup f^{-1}(e_\alpha)$. For example we can do this by taking a function $\varphi : I^{i-1} \to I$ supported near $\pi(f^{-1}(q))$ and pushing its graph $\Gamma_\varphi \subset I^i$ by the straight down isotopy outside of I^i (and pushing anything underneat to the outside).



Proof proceeds by induction on $A_k \subset A$. $A_k = C+$ cells of dim $< k$. $X_k = A_k \cup B$. By conituing this process we lift maps from (D^i, S^{i-1}) to (X, B), maps to (A, C)

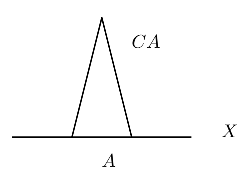
$$\begin{tikzcd}
(D^i, S^{i-1}) \arrow[r, "f"] \arrow[dr] & (A, C) \arrow[d] \\
& (X, B)
\end{tikzcd}
\qquad
\begin{tikzcd}
D^i \arrow[r] \arrow[dr, dashed] & B \\
S^{i-1} \arrow[u, phantom, "\cup"] \arrow[r] & C \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$\square$

Proposition 4.23. *If (X, A) is r-connected and A is s-connected $(r, s \geq 0)$ then*

$$\pi_i(X, A) \underset{\cong}{\longrightarrow} \pi_i(X/A) \qquad \forall i \leq r + s$$

Proof. A s-connected $\implies$ (CA, A) is $(s+1)$-connected, and also given (X, A) r-connected. Then we applying Theorem 4.21 to the pair (X, A), (CA, A), and by using Proposition 1.8 we get the isomorphisms in the diagram below gives the result. $\square$



$$\begin{tikzcd}
\pi_i(X,A) \arrow[r, "\cong"] & \pi_i(X\cup CA, CA) \arrow[r] & \pi_i(X\cup CA/CA) = \pi_i(X/A) \\
& \pi_i(X\cup CA, *) \arrow[u, "\cong"] \arrow[ur, "\cong \text{ collapse}"'] &
\end{tikzcd}$$

By using above we can give another proof of the relative Hurewicz theorem:

Theorem 4.24 (Relative Hurewicz). *If (X,A) is $(n-1)$-connected, $n \geq 2$ and $\pi_1(A) = 0$ then $H_i(X,A) = 0$ for $i \leq n-1$ and $\pi_n(X,A) \cong H_n(X,A)$.*

Proof. Consider the following diagram for $i \leq n$.

$$\begin{tikzcd}
\pi_i(X,A) \arrow[r, "\cong"] & \pi_i(X/A) \arrow[d, "\cong \quad \text{(Absolute Hurewicz)}"] \\
\mathrm{H}_i(X,A) \arrow[r, "\cong"] & \widetilde{H}_i(X/A)
\end{tikzcd}$$

$i \leq (n-1)+1 = n$

□

Corollary 4.25. *If $\pi_1 X = 0$ and $H_i(X) = 0$ for $i > n$, then we have the following: ($Y^{(k)}$ denotes any k-dimensional CW-complex)*

- $X \simeq Y^{(n+1)}$.
- *If $H_n(X)$ is free then $X \simeq Y^{(n)}$.*

Proof. By recalling how CW chains are defined and using the hypothesis we get

- $H_i(X^{(n)}) \xrightarrow{\cong} H_i(X)$ when $i \neq n$
- $H_n(X^{(n)}) \to H_n(X)$ is onto

Hence $H_i(X, X^{(n)}) = 0 \;\; \forall\, i \neq n+1$, so by Hurewicz $\pi_{n+1}(X, X^{(n)}) \cong H_{n+1}(X, X^{(n)})$. Also this gives the following short exact sequence (and clearly $H_n(X^{(n)})$ is free)

$$0 \to H_{n+1}(X, X^{(n)}) \longrightarrow H_n(X^{(n)}) \xrightarrow{i_*} H_n(X) \to 0$$

Then we attach $(n+1)$-cells to $X^{(n)}$ kill kernel of i_*. Let $Y^{(n+1)}$ = resulting complex. If $H_n(X)$ is free, then i_* is already an isomorphism, we take $Y^{(n)} = X^{(n)}$. Then Whitehead theorem gives the required homotopy equivalences. □

Corollary 4.26. *If $\pi_1 X = 0$, X is an n-dimensional manifold, then $X \simeq X^{(n)}$.*

4.2 Obstruction Theory

Obstruction theory studies the following problem: Given CW-complexes $A \subset X$, and a map $f : A \to Y$, when can we find an extension $\tilde{f} : X \to Y$ of this map f?.

$$\begin{tikzcd} X \arrow[dr, dashed, "\tilde{f}"] & \\ \cup & \\ A \arrow[r, "f"'] & Y \end{tikzcd}$$

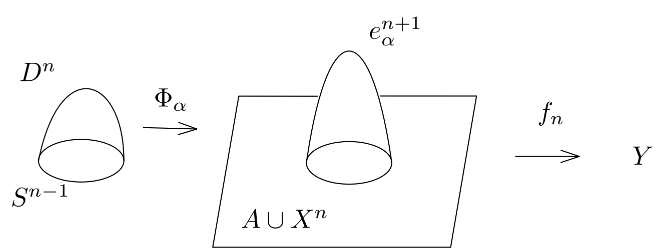
For simplicity we will assume $\pi_1(A) = 0$. We proceed by induction on (relative) skeleton $A \cup X^{(n)}$ of (X, A). Suppose that we have found an extension to n-skeleton ($n \geq 2$):

$$f_n : A \cup X^{(n)} \to Y$$

and we want to extend it to $(n+1)$- skelaton $f_{n+1} : A \cup X^{(n+1)} \to Y$. The answer will be that there is a cohomology class $\theta_n(f) \in H^{n+1}(X, A, \pi_n(Y))$, such that it vanishes if and only if the map f_n can be extended to f_{n+1}, possibly after redefining f_n on the n-cells of $X^{(n)} - X^{(n-1)}$. $\theta_n(f)$ will be the cohomology class of a certain cocycle:

$$\Theta_n(f) \in C^{n+1}(X, A, \pi_n Y) = Hom(C_{n+1}(X, A), \pi_n(Y))$$

Write $X^{(n+1)} = X^{(n-1)} + \sum e_\alpha^{n+1}$, and let Φ_α the characteristic function of e_α^{n+1}. Then define the value of $\Theta(f_n)$ on e_α^{n+1} as follows:



$$\Phi_\alpha = \text{ characteristic map of } e_\alpha^{n+1}$$

$$\left\langle \Theta(f_n), e_\alpha^{n+1} \right\rangle = \Theta(f_n)(e_\alpha^{n+1}) = [f_n \circ \Phi_\alpha]$$

Then extend $\Theta_n(f)$ linearly to make it a homomorphism, and let

$$\Theta_n(f) = \Theta(f_n)$$

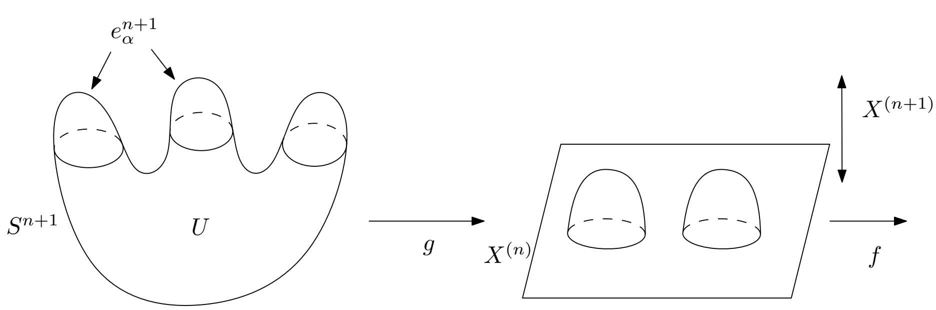
The following properties give the result we want:

Properties of $\Theta_n(f)$:

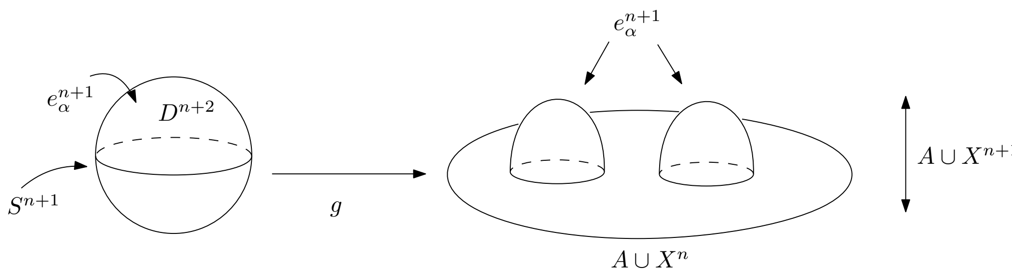
1. $\Theta_n(f)$ only depends on homotopy class of f_n
2. $\Theta_n(f) = 0 \iff f_n$ extends to $f_{n+1} : A \cup X^{n+1} \to Y$
3. $\Theta_n(f)$ is a cocycle i.e. $\delta\Theta_n(f) = 0$. Note that $\delta\Theta_n(f) : C_{n+2}(X, A) \to \pi_n(Y)$
4. If $g : A \cup X^{(n)} \to Y$ with $g_n| = f_n|$ on $A \cup X^{(n-1)}$ then $\Theta(f_n) - \Theta(g_n)$ is coboundary
5. By varying the $[f_n]$ rel $A \cup X^{(n-1)}$, we can change $\Theta(f_n)$ by an arbitrary coboundary
6. $\Theta_n(f)$ is natural with respect to maps $(X, A) \to (X', A')$

Lemma 4.27. *Let (X, A) be a 1-connected CW pair $\pi_1 A = 0$, for $n \geq 2$ $f : S^{n+1} \to A \cup X^{(n+1)}$ with $f_* \{S^{n+1}\} = \sum a_\alpha e_\alpha^{n+1}$. Then $f \simeq g$ with $g^{-1} int(e_\alpha^{n+1}) =$ $|a_\alpha|$-disjoint open balls in S^{n+1} each mapped homeomorphically onto e_α^{n+1}.*

Proof. This is done by homotoping f to a map g transversal to the tips of the cells e_α^{n+1} similar to the argument of Proposition 4.19 $\square$



Properties (**1**), (**2**) and (**6**) trivially hold. Next we prove the property (**3**); The CW-chains $C_{n+2}(X, A)$ are generated by $(n+2)$- cells $e^{n+2} = D^{n+2}$. We apply Lemma 4.27 to the attaching map $g : \partial e^{n+1} = S^{n+1} \to A \cup X^{(n+1)}$ of this cell. This decomposes S^{n+1} into disjoint codimension zero regions $\cup e_\alpha^{n+1} \smile U$, and as a chain $\partial e^{n+2} = \sum a_\alpha e_\alpha^{n+1}$ (here we are abusing notation refering e_α^{n+1} as cells in both S^{n+1} and X).



$$\sum a_\alpha[\partial e_\alpha^{n+1}] = 0 \quad \text{in} \quad \pi_n(A \cup X^n) \quad \text{(because it bounds } U\text{)}$$

$$\begin{aligned}
\left\langle \delta\Theta(f_n), e^{n+2} \right\rangle &= \left\langle \Theta(f_n), \partial e^{n+2} \right\rangle = \left\langle \Theta(f_n), \sum a_\alpha e_\alpha^{n+1} \right\rangle = \\
\sum a_\alpha \left\langle \Theta(f_n), e_\alpha^{n+1} \right\rangle &= \sum a_\alpha [f_n \circ \partial e_\alpha] = (f_n)_\# \sum a_\alpha [\partial e_\alpha] = (f_n)_\#(0) = 0
\end{aligned}$$

Next we check (**4**). Assume $g_n : A \cup X^n \to Y$ agrees with $f_n| : A \cup X^{n-1} \to Y$, since for each e_α^n n-cell in $X - A$, $f_n|_{\partial e_\alpha^n} = g_n|_{\partial e_\alpha^n}$, we can define a map from $S^n = e_\alpha^n \cup -e_\alpha^n$ $f_n - g_n : S^n \to Y$ (f_n on upper hemisphere, g_n on lower hemisphere). This defines

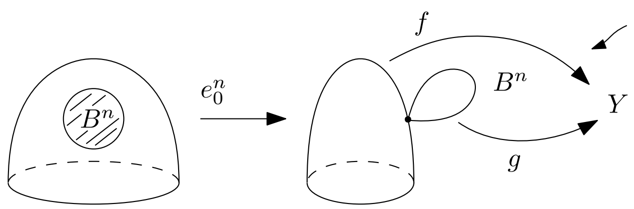
$$d(f_n, g_n) : C_n(X, A) \to \pi_n(Y)$$

We claim $\delta(d(f_n, g_n)) = \Theta(f_n) - \Theta(g_n)$. To see this let e_α^{n+1} be an $(n+1)$-cell in (X, A). We put the attaching map $\partial e_\alpha^{n+1} = S^n \to A \cup X^{(n)}$ into the standard position as described in Lemma 4.27. In particular $f_n = g_n$ on $S^n - \cup e_i^n$, then express ∂e_α^{n+1} as a chain

$$\partial e_\alpha^{n+1} = \sum n_{\alpha\beta} e_\beta^n \quad \text{in} \quad C_n(X, A) \implies$$

$$\begin{aligned}
\left\langle \Theta(f_n), e_\alpha^{n+1} \right\rangle - \left\langle \Theta(g_n), e_\alpha^{n+1} \right\rangle &= [f_n \circ \partial e_\alpha^{n+1}] - [g_n \circ \partial e_\alpha^{n+1}] \\
&= \sum n_{\alpha\beta} \left\langle d(f_n, g_n), e_\beta^n \right\rangle \\
&= \left\langle d(f_n, g_n), \sum n_{\alpha\beta} e_\beta^n \right\rangle \\
&= \left\langle d(f_n, g_n), \partial e_\alpha^{n+1} \right\rangle \\
&= \left\langle \delta d(f_n, g_n), e_\alpha^{n+1} \right\rangle
\end{aligned}$$

For (**5**), we need to show that we can vary $\Theta(f_n)$ by an arbitrary coboundary, by homotopies of f_n rel $A \cup X^{n-1}$. Let e_0^n be an n-cell in (X, A). Let $g \in \pi_n(Y)$ then we claim $f_n \simeq f'_n$ so that $\Theta(f_n) - \Theta(f'_n) = \delta(\mu)$, where $\langle \mu, e_0^n \rangle = g$ and $\langle \mu, e^n \rangle = 0$ for all other e^n



Pick a small ball $B^n \subset e_0^n$, then homotop f_n so that $f_n|_{B^n} = y_0 \in Y$.

$$\text{Then define } f'_n \text{ with } f'_n|_{X^n - \mathrm{int} B^n} = f_n \text{ and}$$

$$f'_n : (B^n, \partial B^n) \to (Y, y_0) \text{ to be } g \in \pi_n(Y, y_0)$$

Clearly $d(f'_n, f_n) = \mu$. Hence by the the property (4) we have:

$$\Theta(f_n) - \Theta(f'_n) = \delta\mu$$

4.3 Some homotopy groups of Steifel Manifolds

4.3.1 Real Steifeld manifold $V_k(\mathbb{R}^n)$

Oriented k frames in $\mathbb{R}^n$ is $V_k(\mathbb{R}^n) = SO(n)/SO(n-k), \;\; n \geq 3, \;\; k \geq 2$

Theorem 4.28.

$$\pi_i(V_k(\mathbb{R}^n)) = \begin{cases} 0 & \textit{if} \quad i < n-k \\ \mathbb{Z}_2 & \textit{if} \quad n-k \;\textit{odd} \\ \mathbb{Z} & \textit{if} \quad n-k \;\textit{even} \end{cases}$$

Lemma 4.29. *The obvious fibration $SO(n) \xrightarrow{i} SO(n{+}1) \xrightarrow{\pi} S^n$ is the principal bundle $P(TS^n)$ associated to the tangent bundle TS^n. Also $V_2(\mathbb{R}^n)$ is the unit sphere bundle of the tangent bundle of $T(S^{n-1})$. In particular we have a fibration $S^{n-2} \to V_2(\mathbb{R}^n) \to S^{n-1}$*

Proof. Let $\{e_1, \ldots, e_n\}$ be a tangent frame at $x_0 = (1, 0, \ldots, 0) \in S^n \subset \mathbb{R}^{n+1}$. Define

$$\begin{array}{llll} \phi : SO(n+1) & \xrightarrow{\cong} & P(TS^n) & \text{Frames on } TS^n \\ \quad\;\; \cup & & & \\ \quad\;\; A & \overset{\phi}{\mapsto} & \{Ae_1, \ldots, Ae_n\} & \text{Frames at } A(x_0) \end{array}$$

ϕ is onto and $1-1$. $SO(n+1)$ acts on $P(TS^n)$ transitively with stabilizer $SO(n)$. The second part follows from the fact that $V_2(\mathbb{R}^{n-1})$ consists of pairs of orthogonal unit vectors in $\mathbb{R}^n$. ∎

Consider the homotopy exact sequence of the bundle $SO(n) \xrightarrow{i} SO(n+1) \xrightarrow{\pi} S^n$

$$\pi_{i+1}(S^n) \;\longrightarrow\; \pi_i(SO(n)) \;\xrightarrow{i_\#}\; \pi_i(SO(n+1)) \;\longrightarrow\; \pi_i(S^n) \;\Longrightarrow$$

$$i_\# : \pi_i(SO(n)) \xrightarrow{\cong} \pi_i(SO(n+1)) \quad \text{when} \;\; i \leq n-2$$

There is a natural inclusion $V_k(\mathbb{R}^n) \longrightarrow V_{k+1}(\mathbb{R}^{n+1})$ giving the commuting diagram:

$$\begin{array}{ccccccc}
\pi_i(SO(n-k)) & \longrightarrow & \pi_i(SO(n)) & \longrightarrow & \pi_i(V_k(\mathbb{R}^n)) & \longrightarrow & \pi_{i-1}(SO(n-k)) \\
\downarrow = & & \downarrow & & \downarrow & & \downarrow = \\
\pi_i(SO(n-k)) & \longrightarrow & \pi_i(SO(n+1)) & \longrightarrow & \pi_i(V_{k+1}(\mathbb{R}^{n+1})) & \longrightarrow & \pi_{i-1}(SO(n-k))
\end{array}$$

$$\therefore \quad \pi_i(V_k(\mathbb{R}^n)) \cong \pi_i(V_{k+1}(\mathbb{R}^{n+1})) \text{ when } i \leq n-2 \tag{4.2}$$

Since $k \geq 2$, then $i < n-k \implies i < n-2$. By iterating (4.2), for $i \leq n-k$ we get

$$\pi_i(V_k(\mathbb{R}^n)) \cong \pi_i(V_{k-1}(\mathbb{R}^{n-1})) \cong \ldots \pi_i(V_{k-(k-2)}(\mathbb{R}^{n-(k-2)})) = \pi_i V_2(\mathbb{R}^{n-k+2})$$

This means, we only need to prove the theorem for $V_2(\mathbb{R}^N)$. Now recall we have

$$\text{the fibrations} \left(\begin{array}{ccccc}
SO(n-1) & & & & \\
\downarrow & & & & \\
SO(n) & \xrightarrow{i} & SO(n+1) & \xrightarrow{\pi} & S^n \\
\downarrow \pi' & & & & \\
S^{n-1} & & & &
\end{array} \right) \implies$$

$$\begin{tikzcd}
{} \arrow[r] & \pi_n(S^n) \arrow[r, "\partial"] \arrow[dr, dashed] & \pi_{n-1}(SO(n)) \arrow[r, "i_\#"] \arrow[d, "\pi'_\#"] & \pi_{n-1}(SO(n+1)) \arrow[r] & {} \\
& & \pi_{n-1}(S^{n-1}) & &
\end{tikzcd}$$

Lemma 4.30.
$$\begin{array}{rclc}
\pi'_\#\partial & = & 0 & \textit{if } n \textit{ odd} \\
\pi'_\#\partial & = & \times 2 & \textit{if } n \textit{ even}
\end{array}$$

Proof. For this, we consider the commuting diagram (the vertical maps are fibrations), and the corresponding homotopy exact sequences (∂ and $\bar{\partial}$ will be the connecting homomorphisms of homotopy exact sequences of the vertical fibrations)

$$
\begin{tikzcd}
SO(n) \arrow[r, "\pi'"] \arrow[d, "i"] & SO(n)/SO(n-1) \arrow[d] & = & S^{n-1} \\
SO(n+1) \arrow[r] \arrow[d, "\pi"] & SO(n+1)/SO(n-1) \arrow[d, "\bar{\pi}"] & = & E \\
S^n \arrow[r] \arrow[u, dashed, bend left, "s"] \arrow[uu, bend left=60, "\partial"] & S^n \arrow[u, dashed, bend left, "s"] \arrow[uurr, bend right=40, "\bar{\partial}"'] & &
\end{tikzcd}
$$

Here $E = V_2(\mathbb{R}^{n+1})$ is the oriented sphere bundle $S^{n-1} \longrightarrow E \xrightarrow{\bar{\pi}} S^n$ of the tangent bundle of S^n (we know when $n = 1, 3, 7$ this bundle is trivial). We need to compute

$$\bar{\partial} = \pi'_{\#}\partial : \pi_i(S^n) \longrightarrow \pi_{i-1}(S^{n-1})$$

If the bundle $\bar{\pi} : E \to S^n$ has a section we have $s : S^n \to E$ with $\pi_{\#} \circ s_{\#} = Id \implies \bar{\pi}_{\#}$ is onto $\implies \bar{\partial} = 0$. Having a section is equivalent to $\chi(S^n) = 0 \iff n$ is odd

$$\pi_n(E) \xrightarrow{\bar{\pi}_{\#}} \pi_n(S^n) = \mathbb{Z} \xrightarrow{\bar{\partial}} \mathbb{Z} = \pi_{n-1}(S^{n-1}) \longrightarrow \pi_{n-1}E \longrightarrow 0$$

$$\implies \pi_{n-1}V_2(\mathbb{R}^{n+1}) = \pi_{n-1}(E) \overset{\text{Hurewicz}}{\cong} \mathrm{H}_{n-1}(E) \overset{\text{PD}}{\cong} \mathrm{H}^n(E) \cong \begin{cases} \mathbb{Z}_2 & \text{if n is even} \\ \mathbb{Z} & \text{if n is odd} \end{cases}$$

The last part is from the Gysin sequence, and $\chi(S^n) = 0$ or 2 depending on parity of n.

$$
\begin{array}{ccccccc}
\to \ \mathrm{H}^0(S^n) & \xrightarrow{\smile\chi} & \mathrm{H}^n(S^n) & \longrightarrow & \mathrm{H}^n(E) & \longrightarrow & 0 \\
\Psi & & \Psi & & & & \\
1 & \rightsquigarrow & \chi(S^n).[S^n] & & & &
\end{array}
$$

Therefore when n is even $\bar{\partial}$ is multiplication by 2. $\square$

Proposition 4.31. *For $i_{\#} : \pi_{n-1}(SO(n)) \longrightarrow \pi_{n-1}(SO(n+1))$ we have*

$$\mathit{keri}_{\#} = \begin{cases} \mathbb{Z} & \mathit{if} \quad n \ \mathit{even} \\ \mathbb{Z}_2 & \mathit{if} \quad n \ \mathit{odd\ and}\ n \neq 1,3,7 \\ 0 & \mathit{if} \quad n = 1,3,7 \end{cases}$$

Proof. : By the previous lemma we know the maps given by dotted arrows:

$$\begin{tikzcd}
 & & \pi_{n-1}(S^{n-1}) & \\
\pi_n(SO(n+1)) \arrow[r, "\pi_\#"] & \pi_n(S^n) \arrow[r, "\partial"] \arrow[ur, dashed] & \pi_{n-1}(SO(n)) \arrow[r, "i_\#"] \arrow[u, "\pi_\#"'] & \pi_{n-1}(SO(n+1)) \\
\pi_{n+1}(S^{n+1}) \arrow[u, "\partial"] \arrow[ur, dashed, "\begin{array}{c}0 \text{ if n is even} \\ \times 2 \text{ if n is odd}\end{array}"'] & & \mathrm{im}\partial = \mathrm{ker} i_\# \arrow[u, phantom, sloped, "\subset"] &
\end{tikzcd}$$

$$\mathrm{ker} i_\# = \mathrm{im}\partial \cong \pi_n(S^n)/\pi_\#(SO(n+1)) \cong \begin{cases} \mathbb{Z} & \text{if n is even} \\ \mathbb{Z}_2 & \text{if n is odd} \end{cases}$$

- If $\pi_\#$ is onto then $\exists \in \pi_n SO(n+1)$ such that

$$\begin{tikzcd}
S^n \arrow[r, "\alpha"] \arrow[rr, dashed, bend right, "\simeq 1"'] & SO(n+1) \arrow[r, "\pi"] & S^n
\end{tikzcd}$$

  Therefore α is a section of of the fibration $SO(n+1) \to S^n$ hence $\Rightarrow n = 1,3,7$

- If $\pi_\#$ is not onto, then there are two cases:

a) If n is odd, then $2\mathbb{Z} \subset \mathrm{im}(\pi_\#) \subsetneq \mathbb{Z} \Rightarrow \mathrm{im}(\pi_\#) = 2\mathbb{Z}$

b) If n is even, then $\pi_\#\partial = \times 2$ (by using the maps ∂ and $\pi_\#$ on the right), hence $\pi_\#\partial$ is $1-1 \Rightarrow \partial$ is one-to-one, then from the diagram we conclude $\mathrm{ker} i_\# = \mathbb{Z}$

$\square$

4.3.2 Complex Steifel manifold $V_k(\mathbb{C}^n)$

Complex k frames in $\mathbb{C}^n$ is $V_k(\mathbb{C}^n)$, it is given by:

$$\begin{aligned} V_k(\mathbb{C}) &= U(n)/U(n-k) \\ V_1(\mathbb{C}) &= U(n)/U(n-1) \cong S^{2n-1} \subset \mathbb{C}^n \end{aligned}$$

The fibration $U(n-1) \to U(n) \xrightarrow{\pi} S^{2n-1}$ gives the Long Exact Sequence

$$\ldots \longrightarrow \pi_{i+1}(S^{2n+1}) \xrightarrow{\partial} \pi_i(U(n)) \longrightarrow \pi_i(U(n+1)) \longrightarrow \pi_i(S^{2n+1}) \longrightarrow \ldots$$

$$\text{Therefore} \quad \pi_i(U(n)) \cong \pi_i(U(n+1)) \text{ when } i \leq 2n-1$$

Similar to real case we have natural inclusions $V_k(\mathbb{C}^n) \longrightarrow V_{k+1}(\mathbb{C}^{n+1})$

$$\begin{array}{ccccc} U(n-k) & \longrightarrow & U(n) & \longrightarrow & V_k(\mathbb{C}^n) \\ \downarrow = & & \downarrow & & \downarrow \\ U(n-k) & \longrightarrow & U(n+1) & \longrightarrow & V_{k+1}(\mathbb{C}^{n+1}) \end{array}$$

$$\begin{array}{ccccccc} \pi_i(U(n-k)) & \longrightarrow & \pi_i(U(n)) & \longrightarrow & \pi_i(V_k(\mathbb{C}^n)) & \longrightarrow & \pi_{i-1}(U(n-k)) \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\ \pi_i(U(n-k)) & \longrightarrow & \pi_i(U(n+1)) & \longrightarrow & \pi_i(V_{k+1}(\mathbb{C}^{n+1})) & \longrightarrow & \pi_{i-1}(U(n-k)) \end{array}$$

$$\begin{array}{rcllr} \pi_i(V_k(\mathbb{C}^n)) & \cong & \pi_i(V_{k-1}(\mathbb{C}^{n-1})) & \text{when} & i \leq 2(n-1)-1 \\ & \vdots & & & \\ \pi_i(V_k(\mathbb{C}^n)) & \cong & \pi_i(V_{k-p}(\mathbb{C}^{n-p})) & \text{when} & i \leq 2(n-p)-1 \\ & \vdots & & & \\ & \cong & \pi_i(V_1(\mathbb{C}^{n-k+1})) \cong \pi_i(S^{2n-2k+1}) & \text{when} & i \leq 2(n-k+1)-1 \end{array}$$

Theorem 4.32. *$V_k(\mathbb{C}^n)$ is $2(n-k)$ connected, and $\pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) \cong \mathbb{Z}$*

Chapter 5

Characteristic classes

Let $V_k^n(X)$ be the isomorphism classes of k^n-vector bundles $E \to X$ over X, where $k = \mathbb{R}$ or $\mathbb{C}$, and denote $V_k(X) = \oplus V_k^n(X)$. Let $\widetilde{V_{\mathbb{R}}^n}(X)$ be the isomorphism class of oriented real vector bundles, and $\widetilde{V_{\mathbb{R}}}(X) = \oplus \widetilde{V_{\mathbb{R}}^n}(X)$, and let $H^*(X;\mathbb{R}) = \oplus H^i(X;\mathbb{R})$.

5.0.1 Steifel-Whitney classes

Definition 5.1. The total **Steifel-Whitney** class is a map:

$$w : V_{\mathbb{R}}(X) \to H^*(X;\mathbb{Z}_2) = \oplus_i H^*(X;\mathbb{Z}_2)$$

where $w(E) = 1 + w_1(E) + w_2(E) + ..$ with $w_i(E) \in H^i(X,\mathbb{Z}_2)$ $i = 0, 1, ..$ satisfying the following four axioms ($w_i(E)$ is called the ith Steifel-Whitney class of E).

(a) Naturality: If $f : X \to Y$ is continuous function then $w(f^*E) = f^*w()E)$

(b) Whitney sum: $w(E \oplus F) = w(E)w(F)$

(c) Finiteness: If $E \in Vec_{\mathbb{R}}^n(X)$ then $w_i(E) = 0$ for $i > n$

(d) Nontriviality: If $L \to \mathbb{RP}^1$ is the tautological line bundle, then $w_1(L)$ is the genertator of $H^1(\mathbb{RP}^1;\mathbb{Z}_2) = \mathbb{Z}_2$

The following proposition says that any two maps $P, Q : V_{\mathbb{R}}(X) \to H^*(X;\mathbb{Z}_2)$, satisfying the axioms (a), (b), (c) above, have to agree if they agree on line bundles. This is because $f^*(PE) = P(f^*E) = Q(f^*E) = f^*(QE)$, with f^* is an injection. Hence the axioms above determine Steifel-Whitney classes.

Proposition 5.2 (Splitting principle)**.** *Given any real bundle $E \to X$, there is some space Y and a map $f : Y \to X$ satisfying:*

- *$f^*(E) = L_1 \oplus ... \oplus L_n$, where each L_i is a line bundle.*
- *$f^* : H^*(X;\mathbb{Z}_2) \to H^*(Y;\mathbb{Z}_2)$ is a monomorphism.*

Proof. The proof is by induction on the dimension of the vector bundle. Let $\pi : E \to X$ be a real vector bundle, and $E_x = \pi^{-1}(x)$ denote the fiber over $x \in X$. Let $p : P(E) \to X$ be the projectivization of E, i.e. $P(E)$ is the bundle whose fibers consists of lines $\{l\}$ through the origin in E_x. More precisely if $P \to X$ the principal frame bundle of E, then $P(E)$ is the associated bundle to the obvious representation $\rho : O(n) \to \mathrm{End}\,(\mathbb{RP}^{n-1})$.

$$P(E) = P \times_\rho \mathbb{P}^{n-1}$$

Then Theorem 3.7 says that $H^*(P(E);\mathbb{Z}_2)$ is a free $H^*(X;\mathbb{Z}_2)$-module with module structure $(x, v) \mapsto p^*(x) \cup v$, in particular p is monic in cohomology. Also the pull-back bundle $p^*E \to P(E)$ splits a line bundle $p^*(E) \cong L \oplus E'$, because the fibers over each $l \in P(E)$ splits as $l \oplus l^\perp \cong E_x$. Now apply induction to the $\mathbb{R}^{n-1}$-bundle $E' \to P(E)$ □

5.0.2 Chern classes

Chern classes are complex version of the Steifel-Whitney classes, they are defined by the complex versions of the same axioms.

Definition 5.3. The total **Chern class** is a map:

$$c : V_{\mathbb{C}}(X) \to H^{2*}(X;\mathbb{Z}) = \oplus_i H^{2i}(X;\mathbb{Z})$$

where $c(E) = 1 + c_1(E) + c_2(E) + ..$ with $c_i(E) \in H^{2i}(X, \mathbb{Z})$, $i = 0, 1, ..$ satisfying:

(a) Naturality: If $f : X \to Y$ is continuous function then $c(f^*E) = f^*c(E)$

(b) Whitney sum: $c(E \oplus F) = c(E)c(F)$

(c) Finiteness: If $E \in Vec^n_{\mathbb{C}}(X)$ then $c_i(E) = 0$ for $i > 2n$

(d) Nontriviality: If $L \to \mathbb{CP}^1$ is the tautological line bundle, then $c_1(L)$ is the negative of the canonical (complex) generator of $H^2(\mathbb{CP}^1; \mathbb{Z}) = \mathbb{Z}$

As in the real case, the following says that Chern classes are uniquely determined by their axioms. If two maps $P, Q : V(X) \to H^*(X; \mathbb{Z})$ satisfy the naturality axioms (a), (b), (c), then they agree if they agree on line bundles.

Proposition 5.4 (Complex splitting principle). *Given any compex bundle $E \to X$, there is some space Y and a map $f : Y \to X$ satisfying:*

- $f^*(E) = L_1 \oplus ... \oplus L_n$, *where each L_i is a complex line bundle.*
- $f^* : H^*(X; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ *is an injection.*

Proof. The proof is similar to the real case, we induct on dimension of E. Given a $\mathbb{C}^n$-vector bundle $\pi : E \to X$, where $E_x = \pi^{-1}(x)$ denotes the fiber over $x \in X$. Let $p : P(E) \to X$ be the projectivization of E, i.e. $P(E)$ is the bundle whose fibers consists of complex lines $\{l\}$ through the origin in E_x. Then by Theorem 3.7 $H^*(P(E); \mathbb{Z})$ is a free $H^*(X : \mathbb{Z})$-module with module structure $(x, v) \mapsto p^*(x) \cup v$, and p is monic in cohomology, and the pull-back bundle $p^*E \to P(E)$ splits a line bundle $p^*(E) \cong L \oplus E'$, as in the real case. Then continue with induction with the $\mathbb{C}^{n-1}$-bundle $E' \to P(E)$ $\square$

5.0.3 Euler class

Definition 5.5. (Section 3.5) The **Euler class** of an oriented real vector bundle $\xi = (E, \pi, X)$: $\mathbb{R}^n \to E \xrightarrow{\pi} X$ is the cohomology class $e = e(\xi) \in \mathrm{H}^n(X)$ defined by

$$e(\xi) = (\pi^*)^{-1} j^*(\tau) = i^* j^*(\tau) \quad \text{where} \quad \tau \text{ is the Thom class}$$

$$\begin{tikzcd}
\tau \in \mathrm{H}^n_c(E) \cong \mathrm{H}^n(DE, SE) \arrow[r, "j^*"] \arrow[d] \arrow[d, dashed, "\cong"'] & \mathrm{H}^n(DE) \arrow[dl, dashed] \arrow[r, bend left, "i^*"] & \mathrm{H}^n(X) \arrow[l, "\pi^*"', "\cong"] \\
\mathrm{H}_0(X) & & e \arrow[u, phantom, "\in"]
\end{tikzcd}$$

Recall Thom isomorphism $\Phi : \mathrm{H}^j(X) \xrightarrow{\cong} \mathrm{H}^{j+n}(E)$, $\Phi(e) = \pi^*(e) \smile \tau \implies$

$$\text{when n is odd} \quad i^*(\Phi(e)) = e \smile e = -e \smile e \implies 2e(\xi) = 0$$

Proposition 5.6. *Let $\xi = (E, \pi, X)$ be an oriented real vector bundle, then*

a) For any map $f : Y \to X$, we have $f^*(e(\xi)) = e(f^*(\xi))$

b) $e(\xi \oplus \eta) = e(\xi) \smile e(\eta)$

c) $[e(\xi)]_2 = w_n(\xi)$, If ξ is a complex bundle then $e(\xi) = c_n(\xi) \in H^{2n}(B; \mathbb{Z})$.

d) $e(\xi) = 0$ if ξ has nowhere zero section

Proof. a) Since $\tau(f^*(\xi)) = f^*(\tau(\xi))$ then result follows from naturality.

b)
$$\begin{tikzcd}
& E_1 \\
E_1 \oplus E_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
& E_2
\end{tikzcd}
\qquad
\begin{array}{l}
\tau(E_i) \in \mathrm{H}^n(E_i, \dot{E}_i) \;\; i = 1, 2 \\
\tau(E_1) \times \tau(E_2) = \pi_1^*(\tau(E_1)) \smile \pi_2^*(\tau(E_2))
\end{array}$$

$$\begin{array}{c}
\mathrm{H}^n(E_1 \oplus E_2, \dot{E}_1 \oplus E_2) \otimes \mathrm{H}^m(E_1 \oplus E_2, E_1 \oplus \dot{E}_2) \\
\downarrow\smile \\
\mathrm{H}^{n+m}(E_1 \oplus E_2, (E_1 \dot{\oplus} E_2)
\end{array}$$

$\tau(E_1) \times \tau(E_2)$ restricts to generator on each fiber which is $\tau(E_1 \oplus E_2)$

c) Assume it is true for line bundles, then use the splitting principle Proposition 5.4;

$$\begin{array}{ccccc} L_1 \oplus \cdots \oplus L_n & = & f^* & \dashrightarrow & E \\ & & \downarrow & & \downarrow \\ & & Y & \xrightarrow[f]{} & X \end{array}$$

$$\begin{aligned} f^*(e(E)) &= e(f^*(E)) = e(L_1 \oplus \cdots \oplus L_n) = e(L_1) \smile \cdots \smile e(L_n) \\ &= w_1(L_1) \smile \cdots \smile w_1(L_n) = w_n(L_1 \oplus \cdots \oplus L_n) \\ &= w_n(f^*(E)) = f^*(w_n(E)) \end{aligned}$$

$$\text{Since } f^* \text{ is monic, we conclude } e(E) = w_n(E) \mod 2$$

It remains to check that for line bundles $[e(\xi)]_2 = w_1(\xi)$. It suffices to check this in the universal line bundle $\xi_1 \to \mathbb{RP}^\infty = K(\mathbb{Z}_2, 1)$. Take the classifying map g of the nontrivial line bundle over the circle (Mobius band) $\gamma \to \mathbb{RP}^1$, i.e. $\gamma = g^*\xi_1$

$$\begin{array}{ccc} \gamma & \longrightarrow & \xi_1 \\ \downarrow & & \downarrow \\ \mathbb{R}P^1 & \xrightarrow{g} & \mathbb{R}P^\infty \end{array}$$

Since $g^* : H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \to H^1(\mathbb{RP}^1; \mathbb{Z}_2)$ is $1-1$ it suffices to check the claim for γ.

d) If there is a nonzero section, we have a section s of the sphere bundle $S(E) \to X$

$$\begin{tikzcd} S(E) \arrow[r, "\pi"'] & B \arrow[l, bend right, "s"'] \end{tikzcd} \qquad \begin{array}{l} \pi \circ s = id, \quad s^* \circ \pi^* = 1 \\ \implies \pi^* \text{ is injective} \implies e = 0 \end{array} \quad ,$$

$$\begin{tikzcd} \mathrm{H}^n(DE, SE) \arrow[r, "j^*"] & \mathrm{H}^n(DE) \arrow[r, "i^*"] \arrow[dr, bend right, "\cong\ i_0^*"'] & \mathrm{H}^n(SE) \arrow[d, "s^*"'] \\ & & \mathrm{H}^n(B) \arrow[u, bend right, "\pi^*"'] \end{tikzcd} \qquad \begin{tikzcd} \tau \arrow[r, "j^*"] & \tau \arrow[r, "i^*"] \arrow[d] & \pi^*(e) \\ & e \arrow[ur, dashed] & \end{tikzcd}$$

□

5.0.4 Pontryagin classes

Definition 5.7. The total **Ponryagin class** is a map:

$$p : \widetilde{V^n_{\mathbb{R}}}(X) \to H^{4*}(X;\mathbb{Z}) = \oplus_i H^{4i}(X;\mathbb{Z})$$

where $p(E) = 1 + p_1(E) + p_2(E) + ..$ with $p_i(E) \in H^{4i}(X, \mathbb{Z})$, $i = 0, 1, ..$ defined by:

$$p_i(E) = (-1)^i c_{2i}(E_{\mathbb{C}})$$

where $E_{\mathbb{C}} = E \otimes \mathbb{C}$ is the complexification of the oriented bundle $E \to X$

Proposition 5.8. *Let* $\mathbb{R}^n \to E \to B$ *be a real oriented bundle, then*

a) $[p_i(E)]_2 = w_{2i}(E)^2$

b) $p_n(E) = e(E)^2$ *(independent of choice of orientation since it is squared)*

Proof. Proof basically follows from Proposition 5.6 and splitting principle, e.g.

$$[p_i(E)]_2 = [c_{2i}(E_{\mathbb{C}})]_2 = w_{4i}(E \oplus E) = w_{2i}(E)^2 \qquad \square$$

Consider the basic bundle operation, which we can perform to a given vector bundle: *Complexify* $\to$ *Forget complex structure* $\to$ *Forget orientation*. We can easily check how a bundle changes after going around this triangle once.

$$\begin{tikzcd} & \text{Complex} \arrow[dr] & \\ \text{Unoriented} \arrow[ur, "\mathbb{C}"] & & \text{Oriented} \arrow[ll] \end{tikzcd}$$

If V is a complex vector space then $V \underset{\mathbb{R}}{\otimes} \mathbb{C} \cong V \oplus \bar{V}$, the identification is given by

$$x \mapsto (i \otimes x + 1 \otimes ix) \oplus (i \otimes x - 1 \otimes ix) \text{ and check :}$$

$$ix \mapsto (i \otimes ix - 1 \otimes x) \oplus (i \otimes ix + 1 \otimes x) = i(i \otimes ix + i \otimes x) \oplus -i(-1 \otimes ix + i \otimes x)$$

Any complex n-bundle E underlies a $2n$-dimensional oriented real bundle. Also $E_{\mathbb{C}} = \mathbb{C} \otimes E \cong E \oplus \bar{E}$ (i.e. first forget complex structure then complexify) has the canonical orientation as complex bundle. We can compare these two orientations of $E_{\mathbb{C}}$ by comparing basis elements:

$$v_1, iv_1, \dots v_{2n}, iv_{2n} \dashrightarrow v_1, \dots, v_{2n}, iv_1, \dots, iv_{2n}$$

They differ by $(-1)^{(2n-1)+2n+\cdots+1} = (-1)^{(2n-1)2n/2} = (-1)^{(2n-1)n} = (-1)^n$. So we get

$$p_n(E) = (-1)^n c_{2n}(E_{\mathbb{C}}) = (-1)^n e(E_{\mathbb{C}}) = e(E \oplus \bar{E}) = e(E)^2$$

Consider conjugate bundle $\bar{\xi} = Hom(\xi, \mathbb{C})$. Here the map $v \mapsto \langle ., v \rangle$ defines conjugate isomorphism $\xi \cong \bar{\xi}$. Then $c_k(\bar{\xi}) = (-1)^k c_k(\xi)$. It suffices to check this on line bundles.

Proposition 5.9. *(Splitting principle for oriented real bundles) Given any oriented real bundle $E \to X$, there is some space Y and a map $f : Y \to X$ satisfying:*

- *$f^*(E) = L_1 \oplus \dots \oplus L_r$, where each L_i is a realification of a complex line bundle (i.e underlying real bundle of a complex line bundle), and $r = [\frac{n}{2}]$*
- *$f^* : H^*(X; \mathbb{Q}) \to H^*(Y; \mathbb{Q})$ is an injection,*

Proof. Let $p : \tilde{G}_2(E) \to X$ be the bundle oriented 2-planes in E, with fibers $p^{-1}(x)$ consisting of oriented 2-planes $\{l\}$ in E_x. If $P \to X$ the principal $SO(n)$ frame bundle of E, $\tilde{G}_2(E)$ is the associated bundle to the obvious representation $\rho : SO(n) \to \text{End } (\tilde{G}_2(\mathbb{R}^n))$.

$$\tilde{G}_2(E) = P \times_\rho \tilde{G}_2(\mathbb{R}^n)$$

From the proof of Theorem 5.14 (c) $H^*(\tilde{G}_2(\mathbb{R}^n); \mathbb{Q}) = \mathbb{Q}(u)/u^n = 0$ for some class $u \in H^2(\tilde{G}_2(E); \mathbb{Q})$. Then from Theorem 3.7 $H^*(G_2(E); \mathbb{Q})$ is a free $H^*(X; \mathbb{Q})$-module with module structure $(x, v) \mapsto p^*(x) \cup v$, in particular p is monic in cohomology. The pull-back bundle $p^*E \to \tilde{G}_2(E)$ splits an oriented 2-plane bundle $p^*(E) \cong L \oplus E'$, because the fibers over each $L \in \tilde{G}_2(E)$ splits as $L \oplus L^\perp \cong E_x$. Now apply induction to the $\mathbb{R}^{n-2}$-bundle $E' \to \tilde{G}_2(E)$

□

5.0.5 Wu Classes

Definition 5.10. Let $\mathbb{R}^k \to E^{n+k} \to M^m$ be a vector bundle on a smooth manifold M.

Recall Thom isomorphism and Steenrod squaring operation

:
$$\mathrm{H}^i(M) \xrightarrow[\cong]{\Phi} \mathrm{H}^{i+k}(E, \dot{E}) \xleftarrow{Sq^i} \mathrm{H}^k(E, \dot{E})$$

By using this Wu defined Steifel-Whitney class by the following formula:

$$w_i(E) = \Phi^{-1} Sq^i(\tau)$$

where $\tau \in \mathrm{H}^k(E, \dot{E})$ is the Thom class. Therefore this cefinition is equivalent to:

$$\pi^* w_i(E) \smile \tau = Sq^i(\tau)$$

We can verify that this definition of Steifel-Whitney classes agree with the original definition by checking the axioms. For example we can $w_k(\xi) = e \pmod 2$ as follows

$$\Phi(w_k) = \pi^*(w_k) \smile \tau = Sq^k(\tau) = \tau^2 = \pi^* i^*(\tau) \smile \tau = \pi^*(e) \smile \tau = \Phi(e)$$

Also for the product bundles $\xi \times \eta \to X \times X$, Thom class can be written as $\tau = \tau_\xi \times \tau_\eta$. Also $Sq(\tau) = Sq(\tau_\xi) \times Sq(\tau_\eta)$. From this we deduce the product formula:

$$(w(\xi) \times w(\eta)) \smile (\tau_\xi \times \tau_\eta) = Sq(\tau_\xi \times \tau_\eta) \implies$$
$$(w(\xi) \smile \tau_\xi) \times (w(\eta) \smile \tau_\eta) = Sq(\tau_\xi) \times Sq(\tau_\eta) \implies$$
$w(\xi \times \eta) = w(\xi) \times w(\eta)$ By restricting to diagonal we get $w(\xi \oplus \eta) = w(\xi) w(\eta)$.

Proposition 5.11. *$d^! w_q(M) = Sq^q(\tau)$, where $d : M \hookrightarrow M \times M$ is the diagonal map*

Proof. Recall Section 3.5 and Example 3.5. $D(\Delta) = \tau$. The normal bundle $\nu = \nu_M$ of the diagonal $\Delta \subset M \times M$ isomorphic to the tangent bundle of M, so $w(M) = d^* w(\nu)$.

$$
\begin{aligned}
Sq^q(\tau) \frown [M \times M] &= (\pi^* w_q \smile \tau) \frown [M \times M] = \pi^* w_q \frown (\tau \frown [M \times M]) \\
&= \pi^* w_q \frown d_*[M] = d_*(d^* \pi^* w_q \frown [M]) \\
&= d_*(w_q(M) \frown [M]) \ \ (\text{since } \ \pi \circ d = Id\ ) \quad \square
\end{aligned}
$$

Definition 5.12. The total **Wu class** of a closed smooth manifold M is defined as $v(M) = 1 + v_1 + v_2 + ..$ with $v_i \in H^i(X, \mathbb{Z}_2)$, $i = 0, 1, ..$ such that for any $x \in H^*(X, \mathbb{Z}_2)$:

$$\langle\ v_i \smile x\ ,\ [M]\ \rangle = \langle\ Sq^i(x)\ ,\ [M]\ \rangle$$

Let $w(M) = 1 + w_1 + w_2 \ldots$ be total Steifel-Whitney class, and let $Sq(x) = x + Sq^1(x) + Sq^2(x) \ldots$ denote the total Steenrod square, then we have:

Theorem 5.13. *(Wu)* $\quad w(M) = Sq(v(M))$

Proof. First of all the tangent bundle of $TM \to M$ is identifiied by the normal bundle N of the diagonal $\Delta \subset M \times M$, and the bundle projection $\pi : N \to M$ is induced from the projection to the first factor, so inparticular $\pi^*(w_i) = w_i \times 1$. In calculation below we will be applying (2) of Proposition 2.13, with $\eta = w_i$, $\alpha = \tau$, $\beta = 1$, $b = [M]$ (we ignore signs these are mod 2 calculations). First we calculate $w = Sq(\tau)/[M]$ as follows:

$$
\begin{aligned}
w_i &= w_i \smile [\ \tau/[M]\ ] \ \ (\textit{Lemma } 3.15\ ) \\
&= [\tau \smile (w_i \times 1)\ ]/[M] \ \ (\textit{Proposition } 2.13\ ) \\
&= [\tau \smile \pi^*(w_i)\ ]/[M] = Sq^i(\tau)/[M]
\end{aligned}
$$

By Theorem 3.31 Thom class $\tau = \sum \alpha \times \alpha^\circ$. By Theorem 3.17 and from $x = \Phi^{-1}\Phi(x)$

$$x = \Phi^{-1}(x \frown [M]) = \tau/(x \frown [M]) = \sum \alpha\ \langle\ \alpha^\#, x \frown [M]\ \rangle = \sum \alpha \langle\ x \smile \alpha^\#, [M]\ \rangle$$

By applying this to Wu class: $v = \sum \alpha \langle\ v \smile \alpha^\circ, [M]\ \rangle = \sum \alpha \langle\ Sq(\alpha^\circ), [M]\ \rangle$ we get

$$Sq(v) = \sum Sq(\alpha)\ \langle\ Sq(\alpha^\circ)\ ,\ [M]\ \rangle = \sum \langle\ Sq(\alpha) \times Sq(\alpha^\circ)\ \rangle/[M] = Sq(\tau)/[M] = w \quad \square$$

5.1 Cohomology of classifying spaces

Let $G_n^{\mathbb{C}}$, G_n,$\tilde{G}_n$ denote the Grassmann manifold of k-dimensioinal complex planes in $\mathbb{C}^\infty$, k-dimensonal real planes in $\mathbb{R}^\infty$, and k-dimensional oriented planes in $\mathbb{R}^\infty$, respectively. We can easily identify them by the following symmetric spaces:

$$G_n^{\mathbb{C}} = U(n)/U(k) \times U(n-k)$$

$$G_n = O(n)/O(k) \times O(n-k)$$

$$\tilde{G}_n = SO(n)/SO(k) \times SO(n-k)$$

To see the first one observe that $U(n)$ acts on $G_n^{\mathbb{C}}$ transitively with stabilizer $U(n-k)$ (similar argument for the other two). Let $\xi_n^{\mathbb{C}} \to G_n^{\mathbb{C}}$, $\xi_n \to G_n$, $\tilde{\xi}_n \to \tilde{G}_n$ be the corresponding universal (tautological) vector bundles, and let $c_i = c_i(\xi_n^{\mathbb{C}})$, $w_i = w_i(\xi_n)$, $p_i = p_i(\xi_n)$, $\tilde{w}_i = w_i(\tilde{\xi}_n)$, $\tilde{p}_i = p_i(\tilde{\xi}_n)$ be the corresponding Chern and Pontryagin classes.

Theorem 5.14. *(a)* $H^*(G_n;\mathbb{Z}_2) = \mathbb{Z}_2\,[w_1,\dots,w_n]$

(b) $H^*(\tilde{G}_n;\mathbb{Z}_2) = \mathbb{Z}_2\,[\tilde{w}_2,\dots,\tilde{w}_n]$

(c) $H^*(G_n^{\mathbb{C}};\mathbb{Z}) = \mathbb{Z}[c_1,\dots,c_n]$.

(d) All torsion of $H^*(G_n;\mathbb{Z})$ *is order* 2 *and if* T *is the torsion subgroup then*

$$H^*(G_n;\mathbb{Z})/T \cong \mathbb{Z}\,[p_1,\dots,p_k]\,,\ \text{where}\ \ k=[n/2]$$

(e) All torsion of $H^*(\tilde{G}_n;\mathbb{Z})$ *is order* 2 *and if* T *is the torsion subgroup then*

$$H^*(\tilde{G}_n;\mathbb{Z})/T = \begin{cases} \mathbb{Z}\,[\tilde{p}_1,\dots\tilde{p}_k] & \text{if}\ \ n=2k+1 \\ \mathbb{Z}\,[\tilde{p}_1,\dots\tilde{p}_k,e]\,/(e^2=\tilde{p}_k) & \text{if}\ \ n=2k \end{cases}$$

Proof. (**a**) We proceed by induction on n, where $E_n \xrightarrow{\pi} G_n$ is the universal bundle.

There is a homotopy equivalence from the sphere bundle $\phi : S(E_n) \to G_{n-1}$, i.e.

$$S(E_n) = \{(L,v) | v \in L, |v|=1\}\ \ ,\ \text{and}\ \ \phi(L,v) = v^\perp$$

$$\begin{tikzcd}
 & S^{n-1} \arrow[d] & \\
S^\infty \arrow[r] & S(E_n) \arrow[r, "\phi", "\simeq"'] \arrow[d, "\pi"'] & G_{n-1} \\
E_n \arrow[r, dashed] & G_n &
\end{tikzcd}$$

By the Gysin sequence of $S(E_n) \xrightarrow{\pi} G_n$ (the total space is identified with G_{n-1})

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\rho} & \mathrm{H}^i(G_n) & \xrightarrow{\smile\chi} & \mathrm{H}^{i+n}(G_n) & \xrightarrow{\pi^*} & \mathrm{H}^{i+n}(G_{n-1}) & \xrightarrow{\rho} & \mathrm{H}^{i+1}(G_n) & \to & \dots \\
 & & & & \Cup & & \Cup & & & & \\
 & & & & w_j(E_n) & \dashrightarrow & w_j(E_{n-1}) & & & &
\end{array}$$

Claim: The Steifel-Whitney classeds are related by $\pi^*(w_j(E_n)) = w_j(E_{n-1})$

Proof of Claim:
$$\begin{aligned}
\pi^*(E_n) &= \{(L, v, w) \mid L \in G_n, \ \ v, w \in L, |v| = 1\} \\
&\cong L \oplus \phi^*(E_{n-1})
\end{aligned}$$

$\phi^*(E_{n-1}) = \left\{(L, v, w) \mid w \in v^\perp\right\}$, and $L = \{L, v, tv\} \to S(E_n)$ (trivial line bundle)

$$\begin{aligned}
\therefore \ \ \pi^*(w_j(E^n)) &= w_j(L \oplus \phi^*(E_{n-1})) \\
&= w_j(\phi^*(E_{n-1})) \\
&= \phi^*(w_j(E_{n-1})) \\
&= w_j(E_{n-1}) \ \ (\phi^* \text{ is an isomorphism})
\end{aligned}$$

By induction we have $\mathrm{H}^*(G_{n-1}) = \mathbb{Z}_2\left[w_j(E_{n-1}) | j \leq n-1\right]$. Also since $\pi^*(w_j(E_n)) = w_j(E_{n-1})$, π^* is onto, hence we have the short exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^i(G_n) & \xrightarrow{\smile\chi} & \mathrm{H}^{i+n}(G_n) & \xrightarrow{\pi^*} & \mathrm{H}^{i+n}(G_{n-1}) & \longrightarrow & 0 \\
 & & & & \omega_j(E_n) & \dashrightarrow & \omega_j(E_{n-1}) & &
\end{array}$$

Also here the map $\mathrm{H}^0(G_n) \to \mathrm{H}^n(G_n)$ is given by $1 \mapsto \chi = w_n(E_n)$. Notice that

$w_n(E_n) \neq 0$, otherwise by universality we would have $w_n = 0$ for all n-dim vector bundles, but this is not the case, e.g. $T(\mathbb{RP}^n) \oplus \epsilon^1 = \xi_1 \underbrace{\oplus \cdots \oplus}_{n+1} \xi_1$ and $w(\mathbb{RP}^n) \neq 0$

Now by induction on n we can show $\forall\ \theta \in \mathrm{H}^k(G_n)$ can be expressed uniquely by polynomials in $\omega_1(E_n), \ldots, \omega_n(E_n)$: By induction $\pi^*\theta = f(\omega_1(E_{n-1}), \ldots, \omega_{n-1}(E_{n-1}))$ for some polynomial f. So by above claim $\theta - f(\omega_1(E_n), \ldots, \omega_n(E_n)) \in im(\pi^*) \implies \theta - f(\omega_1(E_n), \ldots, \omega_n(E_n)) = \xi \smile \omega_n$ for some $\xi \in \mathrm{H}^{k-n}(G_n)$, and such ξ is unique since the map $\smile \omega_n$ is injective. By induction ξ is a unique polynomial g in ω_i's

$$\text{Hence} \quad \alpha = f(\omega_1(E_n), \ldots, \omega_{n-1}(E_n)) = \omega_n g(\omega_1(E_n), \ldots, \omega_n(E_n))$$

(**b**) We have the 2-fold connected covering space $S^0 \to \tilde{G}_n \xrightarrow{\pi} G_n$. We claim

$$\pi^* : \mathrm{H}^*(G_n; \mathbb{Z}_2) \longrightarrow \mathrm{H}^*(\tilde{G}_n; \mathbb{Z}_2)$$

is onto and $\ker(\pi^*) = \langle \omega_1 \rangle$, therefore $\mathrm{H}^*(\tilde{G}_n; \mathbb{Z}_2) = \mathbb{Z}_2[\omega_2, \ldots, \omega_n]$. For this we consider Gysin sequence and see that the map $\smile \omega_1$ is injective in all dimensions

$$\begin{array}{ccccccccccc} 0 & \to & \mathrm{H}^0(G_n) & \xrightarrow[\cong]{\pi^*} & \mathrm{H}^0(\tilde{G}_n; \mathbb{Z}_2) & \xrightarrow{\rho} & \mathrm{H}^0(G_n; \mathbb{Z}_2) & \xrightarrow[1-1]{\smile \chi} & \mathrm{H}^1(G_n; \mathbb{Z}_2) & \to & \mathrm{H}^1(\tilde{G}_n) \\ & & \wr\| & & \wr\| & & & & & & \\ & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & & \chi = \omega_1 & & & \end{array}$$

(**c**) $\exists$ Fibration from the sphere bundla $\phi : S(\xi_n) \to G^{\mathbb{C}}_{n-1}$, $\phi(v, L) = v^\perp$

$$\begin{tikzcd} & S^{2n-1} \arrow[d] & \\ S^\infty \arrow[r] & S(\xi^{\mathbb{C}}_n) \arrow[r, "\phi", "\simeq"'] \arrow[d, "\pi"'] & G^{\mathbb{C}}_{n-1} \\ & G^{\mathbb{C}}_n & \end{tikzcd}$$

By applying Gysin sequence when $i < 2n-1$ we get

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}^{i-2n}(G_n^{\mathbb{C}}) & \xrightarrow{\smile\chi} & \mathrm{H}^i(G_n^{\mathbb{C}}) & \xrightarrow{\pi^*} & \mathrm{H}^i(S(\xi_n^{\mathbb{C}})) & \longrightarrow & \mathrm{H}^{i-2n+1}(G_n^{\mathbb{C}}) & \longrightarrow & \ldots \\
 & & \shortparallel & & c_i(\xi_n) & \dashrightarrow & c_i(\xi_{n-1}) & & \shortparallel & & \\
 & & 0 & & & & & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^0(G_n^{\mathbb{C}}) & \xrightarrow{\smile\chi} & \mathrm{H}^{2n}(G_n^{\mathbb{C}}) & \xrightarrow{\pi^*} & \mathrm{H}^{2n}(S(\xi_n^{\mathbb{C}})) & \longrightarrow & 0 \\
 & & & & \Cap & & \wr\| & & \\
 & & 1 & \dashrightarrow & c_n & & G_{n-1}^{\mathbb{C}} & &
\end{array}$$

For any $\alpha \in \mathrm{H}^i(G_n^{\mathbb{C}})$ we can write $\pi^*(\alpha) = f(c_1(\xi_{n-1}), \ldots, c_{n-1}(\xi_{n-1})$ (induction)

$$\pi^*(\alpha) = f(c_1(\xi_{n-1}), \ldots, c_{n-1}(\xi_{n-1}) \text{ (by induction)}$$

Define $\tilde{\alpha} = f(c_1(\xi_n), \ldots, c_{n-1}(\xi_n)) \implies \pi_0^*(\alpha - \tilde{\alpha}) = 0 \implies \alpha = \tilde{\alpha} + c_n \smile \beta$

By induction on dimension i, we have $\beta = g(c_1(\xi_n), \ldots, c_n(\xi_n))$, so we can write

$$\alpha = F(c_1(\xi_n), \ldots, c_n(\xi_n))$$

(**c**): Let $R = \mathbb{Z}\left[\frac{1}{2}\right] =< \frac{n}{2^i} >\subset \mathbb{Q}$. We now change coefficients from $\mathbb{Z}$ to R, which kills $2-$torsion, and $\mathbb{Z}-$ summands become R summands, so one need to show:

$$\mathrm{H}^*(\tilde{G}_n : R) = \begin{cases} R\,[\tilde{p_1}, \ldots, \tilde{p_k}] & for \quad n = 2k+1 \\ R\,[\tilde{p}_1, \ldots, \tilde{p}_k, e,] & for \quad n = 2k \end{cases}$$

As before we identify the sphere bundle $S(\xi_n) \simeq \tilde{G}_{n-1}$, and use Gysin sequence.

$$\begin{array}{ccc}
S^{n-1} & & \\
\downarrow & & \\
S(\xi_n) & \xrightarrow[\simeq]{\phi} & \tilde{G}_{n-1} \\
\pi\downarrow & & \\
\tilde{G}_n & &
\end{array}$$

$$\begin{array}{ccccccccccc} \dots & \to & \mathrm{H}^i(\tilde{G}_n) & \xrightarrow{\smile\chi} & \mathrm{H}^{i+n}(\tilde{G}_n) & \xrightarrow{\pi^*} & \mathrm{H}^{i+1}(\tilde{G}_{n-1}) & \xrightarrow{\rho} & \mathrm{H}^{i+1}(\tilde{G}_n) & \to & \dots \\ & & & & p_i(\tilde{\xi}_n) & \overset{\pi^*}{\dashrightarrow} & p_i(\tilde{\xi}_{n-1}) & & & & \end{array}$$

- If $n = 2k$ by induction $\mathrm{H}^*(\tilde{G}_{n-1}) = R\,[\tilde{p}_1, \dots \tilde{p}_{k-1}]$ then π^* onto, and we get a short exact sequence, and the proof follows from the $\mathrm{H}^*(G_n; \mathbb{Z}_2)$ model.

- If $n = 2k+1 \implies$ Euler class $e = 0 \in \mathrm{H}^n(\tilde{G}_n, R)$ since e with $\mathbb{Z}-$ coefficients has order 2, then Gysin sequence splits into short exact sequence (π^* is $1-1$).

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathrm{H}^{i+n}(\tilde{G}_n) & \xrightarrow{\pi^*} & \mathrm{H}^{i+n}(\tilde{G}_{n-1}) & \longrightarrow & \mathrm{H}^{i+1}(\tilde{G}_n) & \longrightarrow & 0 \\ & & & & \shortparallel \text{ (induction)} & & & & \\ & & & & R\,[\tilde{p_1}(\xi_{n-1}), \dots, \tilde{p_k}(\xi_{n-1}), e] & & & & \end{array}$$

We need to show $R-$modules $R\,[\tilde{p}_1, \dots, \tilde{p}_k] \subset \mathrm{im}(\pi^*)$ have the same rank.

$$\left.\begin{cases} r_j & = \text{rank of } R\,[\tilde{p}_1, \dots, \tilde{p}_k] \text{ in dim } j \\ r'_j & = \text{rank } \mathrm{H}^j(\tilde{G}_n) \end{cases}\right\} r'_j \geq r_j$$

$R\,[\tilde{p}_1, \dots, \tilde{p}_{k-1}, e]$ is a free $R\,[\tilde{p}_1, \dots, \tilde{p}_k]$ module with basis $\{1, e\}$. Therefore rank of $R\,[\tilde{p}_1, \dots, \tilde{p}_{k-1}, e]$ in dim j is $r_j + r_{j-2k}$. Also from the above exact sequence rank of $R\,[\tilde{p}_1, \dots, \tilde{p}_{k-1}, e] = r'_j + r'_{j-2k}$. Since $r'_m \geq r_m\ \forall m$ then we have $r'_j = r_j$.

(**c**): Let $\varphi : \tilde{G}_n \longrightarrow G_n$ is the 2-fold covering (forgetful) map. and $T : \tilde{G}_n \to \tilde{G}_n$ be the covering transformation. $\varphi^* : \mathrm{H}^*(G_n; R) \to (\tilde{G}_n; R)$ is epimorphism when n is odd, since $\varphi^* p_i(\xi_n) = p_i(\varphi^*\xi_n) = p_i(\tilde{\xi}_n)$. It is also easy to check that, when n even $T^*(e) = -e$. Then the proof follows from the following Claim:

Claim Let $\varphi : \tilde{X} \to X$ be 2-fold covering space and $T : \tilde{X} \to \tilde{X}$ be its covering transformation. Then there is a map $\theta : \mathrm{H}^*(\tilde{X}; R) \to \mathrm{H}^*(X; R)$, with $\theta \circ \varphi^* = id$. In particular φ^* is injective, and $im(\varphi^*) =$ Fixed point set of T^*.

Proof: First define θ in chain level as $\sigma \mapsto 1/2(\sigma + T(\sigma))$, then check the claim. $\square$

5.2 Obstruction definition of characteristic classes

Let $\pi : E \to X$ be a $\mathbb{R}^n$- vecor bundle, and $P \to X$ be its princple $O(n)$-bundle (the bundle of frames in E). Then we can define the bundle of k-frames in E

$$V_k(E) = P \times_\rho V_k(\mathbb{R}^n) \to X$$

as the bundle associated bundle to P by the obvious representation $\rho : O(n) \to V_2(\mathbb{R}^n)$. So the fiber of $V_k(E) \to X$ is just $V_k(\mathbb{R}^n)$. Therefore, by obstruction theory we can always find a section of the bundle $V_{n-k+1}(E) \to X$ over the $k-1$ skeleton of $X^{(k-1)}$, since $V_{n-k+1}(\mathbb{R}^n)$ is $k-2$ connected (Theorem 4.28).

Theorem 5.15. *Let $E \to X$ be a real vector bundle over a CW-complex. Then the Steifel-Whitney class $w_k(E) \in H^k(M; \mathbb{Z}_2)$ can be identified with the mod 2 reduction of the obstruction $[o_k(E)]_2$, of sectioning $V_{n-k+1}(E) \to X$ over the k-skeleton $X^{(k)}$.*

Similarly, if $\pi : E \to X$ is a $\mathbb{C}^n$-vector bundle, we can define the bundle of k-frames in E: $V_k(\mathbb{C}^n) \to V_k(E) \to X$ with fiber $V_k(\mathbb{C}^n)$, which is just the associated bundle, to the principle $U(n)$-bundle $P \to X$, by the representation $\rho : U(n) \to V_2(\mathbb{C}^n)$. Hence, by obstruction theory, we can always find a section of the bundle $V_{n-k+1}(E) \to X$ over the $2k-1$ skeleton of $X^{(2k-1)}$, since $V_{n-k+1}(\mathbb{C}^n)$ is $2k-2$ connected (Theorem 4.32).

Theorem 5.16. *Let $E \to X$ be a complex vector bundle over a CW-complex. Then the Chern class $c_k(E) \in H^{2k}(M; \mathbb{Z}_2)$ can be identified with the obstruction $o_k(E)$ of sectioning $V_{n-k+1}(E) \to X$ over the $2k$-skeleton of X*

Proof. ([MS]) We will only sketch the proof for the real case. The obstuctions $o_k(E)$ of sectioning bundle $E \to X$ are natural with respect to maps $f : Y \to X$, i.e. $o_k(f^*E) = f^*o_k(E)$, where $f^*(E) \to Y$ is the pull-back bundle. So it suffieces to check the proof for the universal bundle $\xi_n \to G_n$. Then $o_k(\xi_n) \in H^k(G_n : \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \dots w_n]$ is clearly a polynomial in the form $o_k(\xi_n) = P(w_1(\xi_n), \dots w_{k-1}(\xi_n) + \lambda w_k(\xi_n)$ for some $\lambda \in \mathbb{Z}_2$.

Next consider the bundle $\gamma^n = \xi_{k-1} \oplus \epsilon^{n-k+1} \to G_{k-1}$ which can be pulled back from the universal bundle $\xi_n \to G_n$ by a map $f : G_{k-1} \to G_n$ (ϵ^{n-k+1} is the trivial bundle).

$$\begin{tikzcd}
\xi_{k-1} \oplus \epsilon^{n-k+1} \arrow[r] \arrow[d] & \xi_n \arrow[d] \\
G_{k-1} \arrow[r] & G_n
\end{tikzcd}$$

Since γ has $(n-k+1)$ section $o_k(\gamma) = 0$, and also $f^*(w_i(\xi_n) = w_i(\xi_{k-1})$, we get

$$0 = o_k(\gamma) = o_k(f^*(\xi_n)) = f^*(o_k(\xi_n)) = f^*P(w_1(\xi_n), \ldots w_{k-1}(\xi_n)) = P(w_1(\xi_{k-1}), \ldots w_{k-1}(\xi_{k-1}))$$

But since in $H^*(G_{k-1}; \mathbb{Z}_2)$ is a free polynomial algebra, there are no polynomial relations, so the polynomial P must be identically zero. Therefore we must have $o_k(\xi_n) = \lambda w_k(\xi_n)$. So it remains to check $\lambda = 1$. For this we only have to check on simple examples.

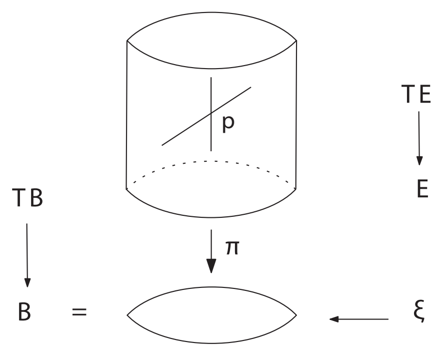
□

5.3 Applications:

- Let $\mathbb{R}^n \longrightarrow E \xrightarrow{\pi} B$ be a vector bundle over a smooth manifold B, then

$$p(TE) = \pi^*(p(E) \smile p(TB))$$

Proof. This is because $TE_p = TF_p \oplus TF_p^{\perp}$, hence $TE \cong \pi^*(\xi) \oplus \pi^*(TM)$



□

- Quoternionic projective $P_n(\mathbb{H})$ is the space of quaternionic lines in $\mathbb{H}^{n+1}$. Consider the tautological line bundle $\gamma_{\mathbb{H}}$: $\mathbb{H} \to P_n(\mathbb{H})$, and let $\gamma_{\mathbb{C}}$ and $\gamma_{\mathbb{R}}$ be the underlying $\mathbb{C}^2-$ and $\mathbb{R}^4-$ bundles. Take the sphere bundle of $\gamma_{\mathbb{H}}$, and apply Ghysin sequence.

$$\begin{array}{ccccc} S^3 & \longrightarrow & S^{4n+3} & \subset & \mathbb{H}^{n+1} \\ & & \downarrow & & \\ & & B := P_n(\mathbb{H}) & & \end{array}$$

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathrm{H}^{i-1}(S^{4n+3}) & \xrightarrow{\rho} & \mathrm{H}^{i-4}(B) & \xrightarrow[\cong]{\smile c_2} & \mathrm{H}^i(B) & \xrightarrow{\pi*} & \mathrm{H}^i(S^{4n+3}) & \longrightarrow & \dots \\ & & \shortparallel & & & & & & \shortparallel & & \\ & & 0 & & & & & & 0 & & \\ & & i < 4n+4 & & & & & & i < 4n+3 & & \end{array}$$

$$\Longrightarrow \quad \mathrm{H}^*(B) = \mathbb{Z}\,[c_2]\,/c_2^{n+1} = 0 \qquad c_2 \in \mathrm{H}^4(B)$$

$$e(\gamma_{\mathbb{R}}) = c_2(\gamma_{\mathbb{C}}) \in \mathrm{H}^4(P_n(\mathbb{H}))$$

$$\begin{array}{lcl} c(\gamma_{\mathbb{C}}) & = & 1+u\ ,\ \text{for some } u \in \mathrm{H}^4(P_n(\mathbb{H})) \\ p_1(\gamma_{\mathbb{C}}) & = & -c_2(\gamma_{\mathbb{C}} \oplus \bar{\gamma}_{\mathbb{C}}) = -2u \\ c(\gamma_{\mathbb{C}} \oplus \bar{\gamma}_{\mathbb{C}}) & = & (1+u)^2 = 1+2u+u^2 \\ e(\gamma_{\mathbb{C}}) & = & c_2(\gamma_{\mathbb{C}}) = u \end{array}$$

Proposition 5.17. *For any integers k, l with $k = 2l$ (mod 4) there exists a vector bundle $\xi \longrightarrow P_1(\mathbb{H}) = S^4$, with $p_1(\xi) = ku \;\; e(\xi) = lu$*

Proof. $\{$Vector bundles on $S^4\} = [S^4\ ;\ \tilde{G}_4(\mathbb{R}^\infty)] = \{\pi_3 SO_4 = \mathbb{Z} \oplus \mathbb{Z}\}$

$$\begin{array}{ccc} & & \bar{\xi} \\ & & \downarrow \\ S^4 & \xrightarrow{f} & \tilde{G}_4 \end{array}$$

$$\left.\begin{array}{ccccc} [f] & \mapsto & p_1(f^*\tilde{\xi}) & \in & \mathbb{Z} \\ [f] & \mapsto & e(f^*\tilde{\xi}) & \in & \mathbb{Z} \end{array}\right\} \in \mathrm{H}^4(S^4)\ ,\quad \left\langle p_1(f^*\tilde{\xi}), [S^4] \right\rangle = \left\langle p_1(\tilde{\xi}), f_*\,[S^4] \right\rangle$$

Let f_1 be the tangent bundle and f_2 be the $\tilde{\gamma}_{\mathbb{R}}$ on $S^4 = P_1(\mathbb{H})$, such that

$$\left.\begin{array}{lclclcl} p_1(f_1^*\,\tilde{\xi}) & = & 0 & & e(f_1^*\,\tilde{\xi}) & = & 2u \\ p_1(f_2^*\,\tilde{\xi}) & = & -2u & & e(f_2^*\,\tilde{\xi}) & = & u \end{array}\right\}$$

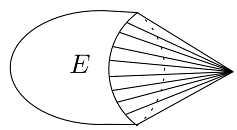
By taking linear combination of f_1 and f_2 we can get f with

$$p_1(f^*\,\tilde{\xi}) = ku, \quad e(f^*\,\tilde{\xi}) = lu \;\text{ where }\; k \equiv 2l\ (mod\ 4)$$

$\Longrightarrow \exists$ vector bundle $\xi \longrightarrow S^4$ with $p_1(\xi) = ku \;\; e(\xi) = lu$ ☐

Example 5.1. *Take the $\mathbb{R}^4$-bundle $\xi \longrightarrow S^4$ with $p_1(\xi) = ku$ and $e(\xi) = u$, and $k = 2$ (mod4). Let E be the disk bundle of ξ. From the Ghysin sequence we get $\partial E \simeq S^7$. We claim when $k = 6$, ∂E is an exotic 7-sphere, i.e. it can not be diffeomorphic to S^7.*

The proof of this claim goes as follows: If ∂E was diffeomorphic to S^7 Thom space T of E would be a smooth 8-manifold. Recall T is obtained from E by coning its boundary.

$T =$ 

We can easily compute, the hmology and the signature of T:

$$\mathrm{H}_i(T) \;=\; \begin{cases} \mathbb{Z} & i = 0,4,8 \\ 0 & otherwise \end{cases} \qquad \Longrightarrow \sigma(T) = \pm 1$$

By Proposition 5.17 $\quad p(T_E) = \pi^*(p(E) \smile p(T_{S^4})) = p_1(T_E) = ku \implies$

$$p_1^2(T_E) = k^2(\text{generetor})$$

Here we used $p(T_{S^4}) = 1$, since by index theorem $p_1(T_{S^4})[S^4] = 3\sigma(S^4) = 0$. Hence

$$1 = \sigma(T) = \frac{7}{45}p_2(T) - \frac{1}{45}p_1^2(T)$$

So $p_2(T) = (45+k^2)/7$, where $k = 2$ $(mod\ 4)$. But when $k = 6$ this is not an integer.

Proposition 5.18. *Let Ω_{4n} be the oriented cobordism ring. Then the signature map $\sigma : \Omega_{4n} \longrightarrow \mathbb{Z}$ is well defined, and it is a ring homomorphism. That is if W^{4n+1} is an oriented compact smooth manifold with $\partial W^{4n+1} = M^{4n}$ then $\sigma(\partial W^{4n+1}) = 0$, and:*

$$\begin{aligned} \sigma(N \times M) &= \sigma(N)\sigma(M) \\ \sigma(N \smile M) &= \sigma(N) + \sigma(M) \end{aligned}$$

Proof. Let $M^{4n} = \partial W^{4n+1}$ then inclusions $M \xhookrightarrow{i} W \xhookrightarrow{j} (W, M)$ give long exact sequence

$$\begin{tikzcd} \longleftarrow \mathrm{H}^{2n+1}(W,\partial W) & \mathrm{H}^{2n}(M) \arrow[l, "\delta"'] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{H}^{2n}(W) \arrow[l, "i^*"'] \arrow[dl, dashed] & \mathrm{H}^{2n}(W,\partial W) \arrow[l, "j^*"'] \leftarrow \\ & \mathrm{im}(i^*)^{\perp} \oplus \mathrm{im}(i^*) & & \end{tikzcd}$$

If $a, b \in \operatorname{im}(i^*) \implies \langle a \smile b, [\partial W]\rangle = 0$. This is because $\delta a = \delta b = 0$ and

$$\langle a \smile b, \partial\,[W]\rangle = \langle \delta(a \smile b), [W]\rangle = \langle \delta a \smile b + a \smile \delta b, [W]\rangle$$

Claim: $\dim(i^*) = \frac{1}{2}\dim \mathrm{H}^{2n}(M^{4n})$ This is because the homology and cohomology groups with $\mathbb{Q}$ coefficients fit into long exact sequences where the vertical maps are the Poincare duality maps:

$$\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
\uparrow & & & & & & & & \downarrow \\
\mathrm{H}^{4n+1}(W,M) & \stackrel{\delta}{\leftarrow} & \mathrm{H}^{4n}(M) & \stackrel{i^*}{\leftarrow} & \mathrm{H}^{4n}(W) & \dots & \mathrm{H}^{2n+1}(W,M) & \stackrel{\delta}{\leftarrow} & \operatorname{coker}(i^*) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathrm{H}^{0}(W) & \stackrel{i^*}{\rightarrow} & \mathrm{H}^{0}(M) & \stackrel{\delta}{\rightarrow} & \mathrm{H}^{1}(W,\partial W) & \dots & \mathrm{H}^{2n}(W) & \stackrel{i^*}{\rightarrow} & \operatorname{im}(i^*) \\
\uparrow & & & & & & & & \downarrow \\
0 & & & & & & & & 0
\end{array}$$

In short we get the following $\mathbb{Q}$-vector spaces with commuting maps:

$$\begin{array}{ccccccccccccc}
0 & \leftarrow & A_1 & \leftarrow & A_2 & \leftarrow & \dots & \leftarrow & A_{6n} & \leftarrow & \operatorname{coker}(i^*) & \leftarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong & & \\
0 & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow & \dots & \rightarrow & B_{6n} & \rightarrow & \operatorname{im}(i^*) & \rightarrow & 0
\end{array}$$

$$\text{Hence } \dim(\operatorname{coker}(i^*)) = \dim(\operatorname{im}(i^*))\ , \text{ since } \operatorname{coker}(i^*) = (\operatorname{im}(i^*))^{\perp}$$

$$\therefore\ \mathrm{H}^{2n}(\partial W) = \operatorname{im}(i^*) \oplus (\operatorname{im} i^*)^{\perp}$$

So, the intersection matrix looks like $2n \times 2n$,with $n \times n$ upper left block entirely consisting of zeros, hence $\sigma(M) = 0$

$$\begin{array}{c} n\{ \\ n\{ \end{array} \left(\begin{array}{c|c} 0 & * \\ \hline * & * \end{array}\right)$$

$\square$

5.3.1 Characteristic Numbers

Steifel-Whitney (or Pontryagin) numbers of a closed smooth manifold M^n are integers obtained evaluating the top dimensional cohomology classes, that are cup products of Steifel-Whitney (or Pontryagin) classes, on the fundamental homology class. We claim that they vanish when M^n is a boundary of a compact smooth manifold $M = \partial W^{n+1}$. We use the cohomology exact sequence of the inclusions $M \overset{i}{\hookrightarrow} W \overset{j}{\hookrightarrow} (W, M)$. The boundary map takes the fundamental class of (W, M) to the fundamental class of M, so

$$\begin{array}{ccc} \mathrm{H}_{n+1}(W, M) & \xrightarrow{\partial} & \mathrm{H}_n(M) \\ [W] & \rightsquigarrow & [M] \end{array}$$

$$\begin{array}{ccccc} \mathrm{H}^n(W) & \xrightarrow{i^*} & \mathrm{H}^n(M) & \xrightarrow{\delta} & \mathrm{H}^{n+1}(W, M) \\ & & \cong \downarrow \frown [M] & & \cong \downarrow \frown [W] \\ & & \mathrm{H}_0(M) & \xrightarrow[i_*]{} & \mathrm{H}_0(W) \end{array}$$

$$\langle \delta v, [M] \rangle = \delta v \frown [W] = i_*(v \frown [M])$$

We can denote the top dimensional cohomology classes that are cup products of Steifel-Whitney classes by $\omega_I(M) = \omega_{i_1}(M) \smile \cdots \smile \omega_{i_s}(M)$, where $I = (i_1, \ldots i_s)$ is a multi-index with $i_1 + \ldots i_s = n$. Note that the characteristic classes of W restrict to charecteristic classes of M, i.e. $i^*\omega_I(W) = \omega_I(M)$. From this we prove the Steifel-Whitney numbers are zero.

$$\langle \omega_I(M), \partial [W] \rangle = \langle \delta \omega_I(M), [M] \rangle = \langle \delta i^* \omega_I(W), [M] \rangle = 0$$

The same argument works for Pontryagin numbers.

5.3.2 Cobordism groups

Consider Thom spaces $MO(k) = T(\xi_k)$, and $MSO(k) = T(\tilde{\xi}_k)$ of the universal bundles

$$\left.\begin{array}{ccccc} E(\xi_k) & \longrightarrow & G_k & = & BO_k \\ E(\tilde{\xi}_k) & \longrightarrow & \tilde{G}_k & = & BSO_k \end{array}\right\}$$

$$T(\xi_k) = E(\xi_k)/\dot{E}(\xi_k) \quad \text{and} \quad T(\tilde{\xi}_k) = E(\tilde{\xi}_k)/\dot{E}(\tilde{\xi}_k)$$

Theorem 5.19 (Thom)**.** *Let η_*, Ω_* be the unoriented and oriented cobordism rings, respectively. Then for $k \geq n+2$ we have identifications:*

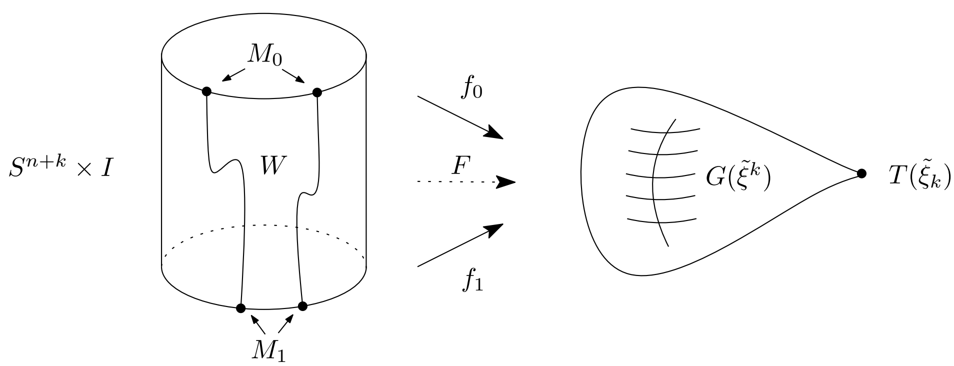
(a) $\eta_n \cong \pi_{n+k}(T(\xi_k))$

(b) $\Omega_n \cong \pi_{n+k}(T(\tilde{\xi}_k))$

Proof. The proofs of (a) and (b) are similar. Define $[f] \mapsto \Phi([f])$ as follows

$$\Phi : \pi_{n+k}(T(\xi_k)) \longrightarrow \eta_n$$

$f : S^{n+k} \longrightarrow T(\xi_k)$ make $f \pitchfork G_k$ then let $M^n = f^{-1}(G_k)$. If $f_0 \simeq f_1 \implies$ the corresponding M_0, M_1 are cobordant. To see this homotope the corresponding homotopy F rel $S^{n+k} \times \mathring{I}$ between f_0 and f_1, so that $F \pitchfork G_k$ then take $W^{n+1} = F^{-1}(G_k)$

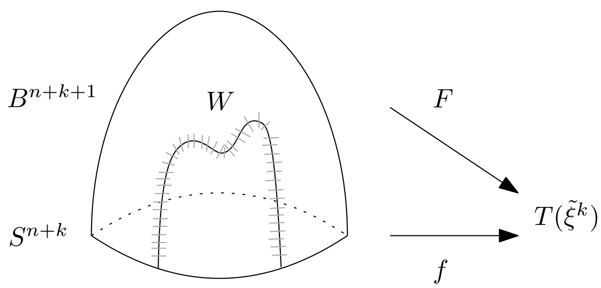


To see Φ **is onto**. Consider the universal bundle $\xi_k^n = E_k(\mathbb{R}^{k+n}) \longrightarrow G_k(\mathbb{R}^n)$. Imbed $M^n \hookrightarrow \mathbb{R}^{n+k}$. Note that when $k \geq n+2$ any two embeddings are isotopic.

$M^n \hookrightarrow \mathbb{R}^{n+k}$. Extend be the normal bundle map $g : \nu(M) \to E(\xi_k^n)$ (with $g \pitchfork 0-$section) to $f : S^{n+k} \to T(\xi_k^n)$. This is done by mapping the complement of the disk bundle $U \subset S^{n+k}$ of $\nu(M)$ to the point of infinity of $T(\xi_k^n)$.

$$\begin{array}{lcc} U \;\cong\; E(\nu_M) & \longrightarrow & E(\xi_k^n) \\ \hookdownarrow & & \downarrow \\ S^{n+k} & \overset{f}{\dashrightarrow} & T(\xi_k^n) \end{array}$$

Φ is $1-1$: If $[f] = 0$, then by making the null homotopy $F : D^{n+k} \to T(\xi_k^n)$ transversal $G_k(\mathbb{R}^n)$ and taking $W = F^{-1}(G_k(\mathbb{R}^n))$ we see that $M = \partial W^{n+1}$.



$W = F^{-1}(\tilde{G}_k(\mathbb{R}^\infty))$

□

Special case of the last theorem is:

Corollary 5.20. *If Ω_n^{fr} = Framed cobordism group $k \geq n+2$*

$$\Omega_n^{fr} = \pi_{n+k}(S^k)$$

5.3.3 Mod $\mathcal{C}$ Homotopy Theory

Let $\mathcal{C}$ be the class of Abelian groups satisfying the following properties:

(1) If $0 \to A' \to A \to A'' \to 0$ short exact sequence of abelian groups then

$$A \in \mathcal{C} \iff \mathcal{A}' \text{ and } \mathcal{A}'' \in \mathcal{C}$$

(2) $A \in \mathcal{C}$ and B is any abelian group $\Rightarrow A \otimes B \in \mathcal{C}$

(3) $A \in \mathcal{C} \Rightarrow \mathrm{H}_n(K(A,1),\mathbb{Z}) \in \mathcal{C}\ \forall\ n > 0$

The following are some classes of abelian groups $\mathcal{C}$ satisfying above property:

- $\mathcal{C}_0 = \{\text{Trivial Group}\}$
- $\mathcal{C}_T = \{\text{Torsion abelian groups}\}$.
- $\mathcal{C}_p = \{\text{Abelian groups with order of every element relatively prime to p}\}$.

Definition 5.21. A homomorhism $f : A \to B$ is called
$\mathcal{C}$- *monomorphism* if $\ker(f) \in \mathcal{C}$
$\mathcal{C}$- *epimorphism* if $\operatorname{coker}(f) \in \mathcal{C}$
$\mathcal{C}$- *isomorphism* if both $ker(f)$ and $\operatorname{coker}(f) \in \mathcal{C}$

$\mathcal{C}$-isomorphism does'nt have inverse in general, to make it an equivalence relation we make the definition: $A \sim B$ if we can find a sequence of abelian groups $A = A_0, A_1, \ldots, A_n = B$, such that either $A_i \to A_{i+1}$ or $A_{i+1} \to A_i$ is an $\mathcal{C}-$isomorphism.

Theorem 5.22 (Hurewicz mod ($\mathcal{C}$))**.** *Let X is* 1- *connected, then*

$$\pi_i(X) \in \mathcal{C} \quad \forall\ i < n \quad \Longrightarrow\ H_i(X) \in \mathcal{C} \quad \forall\ i < n$$

and the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is C-isomorphism.

Theorem 5.23 (Relative Hurewicz mod $\mathcal{C}$)**.** *Given an inclusion $i : A \hookrightarrow X$, such that A is* 1- *connected and the induced map $i_\# : \pi_2(A) \twoheadrightarrow \pi_2(X)$ is onto, then*

$$\pi_i(X,A) \in \mathcal{C} \quad \forall\ i < n \Rightarrow H_i(X,A) \in \mathcal{C}\ \ \forall i < n$$

and the Hurewicz map $h : \pi_n(X,A) \to H_n(X,A)$ is a C-isomorphism.

Theorem 5.24 (Whitehead's mod $\mathcal{C}$). *Let X and Y be 1-connedted, then given a map $f : X \to Y$, such that the induced map $f_\#$ is an isomorphism on π_2, we have*

$$\left\{ \begin{array}{c} f_\# \textit{ is } \mathcal{C}\textit{-isomorphism on } \pi_i \;\forall\, i < n \\ f_\# \textit{ is } \mathcal{C}\textit{- epimorphism on } \pi_n \end{array} \right\} \iff \left\{ \begin{array}{c} f_* \textit{ is } \mathcal{C}\textit{-isomorphism on } H_i \;\forall\, i < n \\ f_* \textit{ is } \mathcal{C}\textit{- epimorphism on } H_n \end{array} \right\}$$

Proposition 5.25. *If X is a CW complex which is $(k-1)-$connected, for $k \geq 2$ then the Hurewicz map*

$$h : \pi_r(X) \to H_r(X, \mathbb{Z}) \textit{ is a } \mathcal{C}_T - \textit{isomorphism for } r < 2k-1$$

Proof. We know $\pi_r S^k \in \mathcal{C}_T \;\; \forall\, r < 2k-1,\; r \neq k$, so we have

- Proposition holds when $X = S^k$

- If Proposition holds for $(k-1)-$connected spaces $X, Y \implies$ holds for $X \vee Y$. To see this we use Hurewicz theorem for pair $(X \times Y, X \vee Y)$ and obtain $\pi_r(X \vee Y) \cong \pi_r(X \times Y) \cong \pi_r(X) \times \pi_r(Y) \;\; \forall r \; \leq 2k-1$.

- If X is $(k-1)-$connected and $\pi_r(X)$ is finitely genereted. Choose a basis for the free part of $\pi_r(X) = \left\langle [S_1^{r_1}], \ldots, \left[S_1^{r_p}\right] \right\rangle$ for all $i < 2k-1$, given by the maps

$$f : Z := S^{r_1} \vee \cdots \vee S^{r_p} \to X$$

By choice the top horizontal map below is $\mathcal{C}_T$-isomrphism for $i < 2k-1$, by the previous case left vertical map below is $\mathcal{C}_T$-isomrphism for $i < 2k-1$. By Whitehead mod $\mathcal{C}_T$ theorem the horizontal map below is mod $\mathcal{C}_T$-isomorphism $\square$

$$\begin{tikzcd} \pi_i(Z) \arrow[r, "\cong"'] \arrow[d] & \pi_i(X) \arrow[d, dashed] \\ \mathrm{H}_i(Z) \arrow[r] & \mathrm{H}_i(X) \end{tikzcd}$$

Corollary 5.26. *Let* $\mathbb{R}^k \to E \to B$ *be a vector bundle then there is a* $\mathcal{C}_T$*-isomorphism:*

$$\pi_{n+k}(T(\xi)) \to H_n(B) \quad \forall\, r < 2k-1$$

Proof. Lrt T be the Thom space. By Thom isomorphism $\mathrm{H}_i(T(\xi)) \xrightarrow{\cong} \mathrm{H}_{i-k}(B)$. By an application Hurewicz theorem T is $(k-1)-$ connected. Hence by the Proposition 5.25

$$\pi_r(T(\xi)) \underset{\mathcal{C}}{\xrightarrow{\cong}} \mathrm{H}_r(T(\xi)) \quad \forall\, r < 2k-1$$

□

By applying this corollary to the universal bundle $\xi_k \to \tilde{G}_k$ we get

Corollary 5.27. *For* $n < k-1$ *we have the following identifications:*

$$\Omega_n \cong \pi_{n+k}(T(\xi_k)) \underset{\mathcal{C}_\mathcal{T}}{\xrightarrow{\cong}} H_n(\tilde{G}_k)$$

$$\Omega_n \otimes \mathbb{Q} \cong H_n(\tilde{G}_k(\mathbb{R}^\infty)) \otimes \mathbb{Q} \cong H_n(\tilde{G}_k(\mathbb{R}^{n+k})) \otimes \mathbb{Q} \cong H_n(BSO(k)) \otimes \mathbb{Q}$$

By appying this to Theorem 5.14 we obtain

Corollary 5.28. Ω_n *is a torsion group for* $n \neq$ *(mod 4), and for* $n =$ *(mod 4) it is a finitely genereted group of rank* $p(r)$ *is the number of partitions of* r*.*

Theorem 5.29. $\Omega_* \otimes \mathbb{Q} = \mathbb{Q}\,[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \dots]$ *(identified as polynomial algebras).*

The signature is a ring homomorphism $\sigma : \Omega_* \to \mathbb{Z}$ (it is zero when $* \neq 0$ mod 4). Also for every partition $I = (i_1, \dots, i_r)$ of $n = i_1 + i_2 + \dots, i_r$ there are maps $p_I : \Omega_{4*} \to \mathbb{Z}$

$$p_I\,[M^{4n}] \;\;=\;\; \langle p_{i_1}(M) \dots p_{i_k}(M), [M^{4n}] \rangle$$

where $p_{i_k}(M) \in H^{4i_k}(M^{4m}; \mathbb{Z})$ are Pontryagin classes. Also $p_I = 0$ when $* \neq 0$ (mod 4)

The signature homomorphisim σ can be expressed as a linear combination of p_I's, $I \in \pi(n)$, where $\pi(n)$ denotes the partitions of n. Let us calculate some special cases

Example 5.2. *(a)* $\sigma(M^4) = \frac{1}{3}\, p_1(M)$

(b) $\sigma(M^8) = \frac{7}{45}\, p_2(M) - \frac{1}{45}\, p_{11}(M)$

Proof. (a) Let $\xi \to \mathbb{CP}^n$ be the tautological line bundle, then $T(\mathbb{C}P^n) \oplus \xi \cong \bar{\xi} \oplus \cdots \oplus \bar{\xi}$ (the sum of $n+1$ copies of $\bar{\xi}$). Hence $p(\mathbb{C}P^2) = (p(\bar{\xi}))^3$. On the other hand

$$c(\xi \oplus \bar{\xi}) = (1+\alpha)(1-\alpha) = 1 - \alpha^2 \implies c_2(\xi \oplus \bar{\xi}) = -\alpha^2$$

$$\text{So by definition } \; p_1(\bar{\xi}) = -c_2(\xi_{\mathbb{C}}) = -c_2(\bar{\xi} \oplus \xi) = \alpha^2$$

$$p(\mathbb{C}P^2) = (p(\bar{\xi}))^3 = (1+\alpha^2)^3 \implies p_1(\mathbb{CP}^2) = 3\alpha^2$$

Then $\sigma(\mathbb{C}P^2) = 1 \Rightarrow \sigma = \frac{1}{3}p_1(M)$. To verify (b) we write

$$\sigma(M^8) = \lambda p^2(M) + \mu p_{11}(M) = \left\langle \lambda p_2(M) + \mu p_1(M)^2, [M] \right\rangle$$

$$p(\mathbb{C}P^4) = (1+\alpha^2)^5 = 1 + 5\alpha^2 + 10\alpha^4 \implies \left\{ \begin{array}{rcl} p_2(M) & = & 10\alpha^4 \\ p_1^2(M) & = & 25\alpha^4 \end{array} \right\}$$

$$\implies 1 = \sigma(\mathbb{C}P^4) = \left\langle \; (10\lambda + 25\mu)\alpha^4 \; , \; \left[\mathbb{C}P^2\right] \right\rangle$$

$$1 = 10\lambda + 25\mu \tag{5.1}$$

$$\begin{array}{rcl} \sigma(\mathbb{C}P^2 \times \mathbb{C}P^2) & = & \sigma(\mathbb{C}P^1) \times \sigma(\mathbb{C}P^2) = 1 \\ p(\mathbb{C}P^2 \times \mathbb{C}P^2) & = & (1+3\alpha^2) \times (1+3\alpha^2) \\ & = & 1 + (3\alpha^2 \times 1 + 1 \times 3\alpha^2) + 9\alpha^2 \times \alpha^2 \end{array}$$

$$\therefore \left\{ \begin{array}{rcl} p_1(\mathbb{C}P^2 \times \mathbb{C}P^2) & = & 1 \times 3\alpha^2 + 3\alpha^2 \times 1 \\ p_2(\mathbb{C}P^2 \times \mathbb{C}P^2) & = & 9\alpha^2 \times \alpha^2 \end{array} \right\}$$

$$p_{11}(\mathbb{C}P^2 \times \mathbb{C}P^2) = \left\langle p_1^2, \left[\mathbb{C}P^2 \times \mathbb{C}P^2\right] \right\rangle = 18$$

$$1 = 9\lambda + 18\mu \tag{5.2}$$

$$\implies \lambda = 7/45 \text{ and } \mu = -1/45$$

□

5.3.4 Natural Classes

Let $q \in \mathbb{Q}[[x]]$ = Ring of power series, An (additive or multiplicative) natural class corresponding q is a map from complex vector bundles to the total cohomolgy :

$$Q : V_{\mathbb{C}}(X) \to \mathrm{H}^*(X, \mathbb{Q}) \quad \text{satisfying :}$$

(a) $f : X \to Y \Rightarrow Q(f^*E) = f^*(QE)$ (naturality)

(b) $Q(E \oplus F) = Q(E) \oplus Q(F)$ (additive), or $Q(E \oplus F) = Q(E)Q(F)$ (multiplicative)

(c) $Q(L) = q(x)$, where $L \to X$ is a complex line bundle and $x = c_1(L)$

Q is **uniquely** defined, and it can be expressedinn terms of chern classes since by the splitting principle we can assume $E = L_1 \oplus \cdots \oplus L_n$ where each L_i complex line bundle $Q(E) = \prod q(x_i)$. Since this expression is summetric in x_i, it can ve expressed in terms of elementary symmetric functions $s_r(x_1, \ldots, x_n)$ in x_i, which are the Chern classes $c_r(E)$. Recall that the total Chern class of $E = L_1 \oplus \cdots \oplus L_n$ is $c(E) = \prod_{i=1}^{n}(1 + x_i)$. So we have

$$Q(E) = \prod q(x_i) = 1 + Q_1(E) + Q_2(E) + \ldots$$

where each $Q_r(E) \in H^{2r}(X; \mathbb{Z})$ is some cup pruducts of chern classes of E.

We can also define natural classes for oriented real vector bundles by

$$Q : \tilde{V}_{\mathbb{R}}(X) \to \mathrm{H}^*(X, \mathbb{Q}) \quad \text{satisfying axioms (a), (b) above and (c') below :}$$

(c') $Q(L) = 1$ if $L \in \tilde{V}^1_{\mathbb{R}}(X)$, and $Q_(L) = q(y)$ if $L \in \tilde{V}^2_{\mathbb{R}}(X)$, where $y = p_1(L)$.

Similar to the complex case, by using the real splitting principle Proposition 5.9, it is enough to know these classes on the real bundles of the form $E = L_1 \oplus \cdots \oplus L_r$, where each L_i is realification of a complex line bundle and $r = [\frac{n}{2}]$. So we get.

$$Q(E) = \prod q(x_i^2) = 1 + Q_1(E) + Q_2(E) + \ldots$$

This is because each $p_1(L_i) = -c_2(L_{\mathbb{C}}) = -c_2(L_i \oplus \bar{L}_i) = x_i^2$, where $x_i = c_1(L_i)$. In particular each $Q_i(E)$ is some cup products of Pontryagin classes of E.

Example 5.3. *From $q = 1 + x \in \mathbb{Q}[[x]]$ we get the total Pontryagin class $E \mapsto p(E)$ as a natural class. For example for $E = L_1 \oplus \cdots \oplus L_r$, we get*

$$p(E) = \prod_{i=1}^{r}(1 + x_i^2) \implies p_i(E) = s_i(x_1^2, \ldots x_r^2)$$

For example $p(T_{\mathbb{CP}^n}) = p(\xi)^{n+1} = (1 + \alpha^2)^{n+1} = q(\alpha^2)^{n+1}$ where α is the generator.

Proposition 5.30. *Any real multiplicative class Q defines a ring homorphism by*

$$\tilde{Q} : \Omega \to \mathbb{Q}$$

$$\tilde{Q}(M) = \langle Q(TM), [M] \rangle$$

This is well defined since Pontryagin numbers are cobordism invariants (Section 5.3.1).

Proof. Write $Q(TM) = 1 + Q_1 + Q_2 + \ldots$ with $Q_k \in H^k(M; \mathbb{Q})$

Using $T(X \times Y) = \pi_1^*(TX) \oplus \pi_2^*(TY)$ $\qquad X \times Y \xrightarrow{\pi_1} X$, $\quad X \times Y \xrightarrow{\pi_2} Y$

$$\begin{aligned} \tilde{Q}(N^n \times M^m) &= \langle Q_{n+m}(TN \times TM), [N] \times [M] \rangle \\ &= \langle Q_n(TN) Q_m(TM), [N] \times [M] \rangle \\ &= \langle Q_n(TN), [N] \rangle \langle Q_m(TM), [M] \rangle \\ &= \bar{Q}(N)\bar{Q}(M) \end{aligned}$$

□

Proposition 5.31. *Let $\pi(n) = \{I = (i_1, \ldots, i_r) \mid i_1 + \ldots i_r = n\}$ be the set of partitions of n, and for each $I \in \pi(n)$ define $M_I = \mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_r}$ Then the set*

$$\{\, [M_I] \mid I \in \pi(n)\} \quad \textit{is linearly independent in } \Omega_* \otimes \mathbb{Q}$$

Proof. It suffices to find collection of linear functions $s_J : \Omega \to \mathbb{Q}$, where $I \in \pi(n)$ such that the matrix whose (I, J) entry $a_{I,J} = s_I([M_J])$ is non singular. Now for each monomial $x_1^{i_1} x_2^{i_2} \dots x_r^{i_r} \in \mathbb{Q}[x_1, \dots, x_n]$ define the symmetric function $\sum_{\sigma \in S(n)} x_{\sigma(1)}^{i_1} x_{\sigma(2)}^{i_2} \dots x_{\sigma(r)}^{i_r}$. Since it is symmetric it can be written in terms of the elemantary symmetric functions

$$S_I(\sigma_1 \dots \sigma_n) = \sum_{\sigma \in S(n)} x_{\sigma(1)}^{i_1} x_{\sigma(2)}^{i_2} \dots x_{\sigma(r)}^{i_r}. \text{ then define}$$

$$s_I(\left[M^{4n}\right]) = \langle S_I(p_1(M), \dots, S_I(p_n(M)), [M] \rangle$$

For example for $T(\mathbb{C}P^{2n}) \oplus \xi \cong \bar{\xi} \oplus \dots \oplus \bar{\xi}$ $(2n+1$ copies of $\bar{\xi})$, we calculate

$$\begin{aligned} p(T\mathbb{C}P^{2n}) &= (1+\alpha^2)^{2n+1} \\ p_i(T\mathbb{C}P^{2n}) &= \sigma_i(\alpha^2, \dots, \alpha^2) \end{aligned}$$

For $I = (n),\ S_I = x_1^n + \dots + x_n^n$. $\quad S_{(n)}\left[\mathbb{C}P^{2n}\right] = 2n+1 \neq 0$ because

$$S_I(x_1, \dots, x_n) = x_1^n + \dots + x_n^n = \alpha^{2n} + \dots + \alpha^{2n} = (2n+1)\alpha^{2n}$$

It is easy to check that $S_I(M \times N) = \sum_{(I_1, I_2) = I} S_{I_1}(M) S_{I_2}(M)$ since

$$\begin{array}{ccccc} I_1 \in \pi(k) & I_2 \in \pi(l) & \Rightarrow & (I_1, I_2) \in \pi(k+l) \\ \shortparallel & \shortparallel & & \shortparallel \\ (i_1, \dots, i_r) & (j_1, \dots, j_s) & & (i_1, \dots, i_r, j_1, \dots, j_s) \end{array}$$

$$\left.\begin{aligned} a = 1 + a_1 + \dots &= (1+x_1) \dots (1+x_n) \\ b = 1 + b_1 + \dots &= (1+x_{n+1}) \dots (1+x_{2n}) \end{aligned}\right\} \text{in } \mathbb{Q}[x_1, \dots, x_{2n}]$$

$$ab = \prod_{i=1}^{2n}(1+x_i), \quad S_I(x_1, \dots, x_n) = S_I(\sigma_1, \dots, \sigma_n)$$

More generally we have $S_I[M_1 \times \dots \times M_s] = \sum_{(I_1, \dots, I_s) = J} S_{I_1}[M_1] \dots S_{I_s}[M_s]$

From this we get when lenght $I <$ lenght J then $a_{I,J} = 0$, and $a_{I,I} \neq 0$

$$a_{I,J} = S_J(\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_r}) = S_J(M_I)$$

$$A_{I,J} = \begin{array}{c} \begin{array}{cccc} \overbrace{\qquad\quad}^{\text{length } 1} & \overbrace{\qquad\quad}^{\text{length } 2} & \overbrace{\qquad\quad}^{\dots} & \overbrace{\qquad\quad}^{\text{length } n} \end{array} \\ \left(\begin{array}{c|c|c|c} * & & & \\ \hline 0 & * & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right) \end{array}$$

□

Signature function $\sigma : \Omega_{4*} \to \mathbb{Q}$ is a ring homorphism, which evaluates 1 on the generators $[\mathbb{CP}^{2k}]$. If we can find a natural $L : \Omega_{4*} \to \mathbb{Q}$ which evaluates 1 on the cobordism generators of $\Omega \otimes \mathbb{Q}$ as the signature does, then it has to agree with the signature. Consequently $\sigma(M^{4n})$ can be expressed in terms of the Ponrtyagin numbers of M. Hirzebruch found the power series $l \in \mathbb{Q}[[]x]]$ which provides such L, namely

$$l(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}}$$

From this we can calculate the corresponding natural class:

$$L(E) = \prod \frac{x_i}{tanh(x_i)} = 1 + L_1(E) + L_2(E) + \dots$$

$$\begin{aligned} L_1(E) &= \frac{1}{3} p_1 \\ L_2(E) &= \frac{1}{45}(7p_2 - p_1^2) \\ L_3(E) &= \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3) \text{ ... etc.} \end{aligned}$$

Theorem 5.32. *(Hirzebruch)* $I(M) = L(M)$

Proof. It suffices to show: $L[\mathbb{CP}^{2n}] = \sigma[\mathbb{CP}^{2n}] = 1$. By using $T(\mathbb{CP}^{2n}) \oplus \varepsilon^1 = \bar{L} \oplus .. \oplus \bar{L}$, where $L \to \mathbb{CP}^{2n}$ is the canonical line bundle, we calculate

$$L[\mathbb{CP}^{2n}] =< (\frac{z}{\tanh\, z})^{2n+1}, \mathbb{CP}^{2n} >$$

where $z \in H^2(\mathbb{CP}^{2n})$ is the generator. Hence $L[\mathbb{CP}^{2n}]$ is the coefficient of z^{2n} in the power series, which we can compute by the residue

$$\frac{1}{2\pi i} \int (\frac{z}{\tanh\, z})^{2n+1} \frac{1}{z^{2n+1}} dz = \frac{1}{2\pi i} \int \frac{dz}{(\tanh\, z)^{2n+1}} = \frac{1}{2\pi i} \int \frac{du}{u^{2n+1}(1-u^2)} = 1$$

where $u = \tanh\, z$ and $dz = du/(1-u^2)$ □

Chapter 6

Spectral sequences

6.1 Exact Couples

An *exact couple* is an exact sequence of Abelian groups is given by

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

where $j \circ i = k \circ j = i \circ k = 0$. Define $d : E \to E$ by $d = j \circ k$ then the map d has the property $d^2 = j \circ (k \circ j) \circ k = j \circ 0 \circ k = 0$ so the homology of the exact couple is

$$E^1 = H(E, d) = (Ker d)/(Im d)$$

Let $D^1 = i(D); E^1 = H(E)$ we can iterate this to construct a new exact couple, called *derived couple*.

$$\begin{tikzcd} D^1 \arrow[rr, "i^1"] & & D^1 \arrow[dl, "j^1"] & & D^n \arrow[rr, "i^n"] & & D^n \arrow[dl, "j^n"] \\ & E^1 \arrow[ul, "k^1"] \arrow[rrrr, dashed, "iterate"] & & & & E^n \arrow[ul, "k^n"] & \end{tikzcd}$$

To be more precise

(1) $i^1 : D^1 \to D^1; i^1 = i_{|D^1}$

(2) $j^1 : D^1 \to E^1; j^1(x) = [j \circ i^{-1}(x)]$

(3) $k^1 : E^1 \to D^1; k^1[e] = k(e)$

It is almost straightforward to check that the maps are well defined and the derived couple is exact. So ones we are given an exact couple we construct the sequence of exact couples, in this construction the sequence $E^{n+1} = H(E^n, d^n)$ is called the *spectral sequence* of the exact couple.

In the similar, way we can talk about exact couples of graded abelian groups. For example they appear natural way when there is filitration of spaces

$$\phi = K^{-1} \subset K^0 \subset K^1 \subset \cdots \subset K$$

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} \qquad \begin{array}{rcl} D & = & \{D_{p,q} = \mathrm{H}_{p+q}(K^p)\} \\ E & = & \{E_{p,q} = \mathrm{H}_{p+q}(K^p, K^{p-1})\} \end{array}$$

$$\begin{array}{lclcll} i_{p,q} & : & D_{p,q} \to D_{p+1,q-1} & ; & \mathrm{H}_{p+q}(K^p) & \to \mathrm{H}_{p+q}(K^{p+1}) \\ j_{p,q} & : & D_{p,q} \to E_{p,q} & ; & \mathrm{H}_{p+q}(K^p) & \to \mathrm{H}_{p+q}(K^p, K^{p-1}) \\ k_{p,q} & : & E_{p,q} \to D_{p-1,q} & ; & \mathrm{H}_{p+q-1}(K^{p-1}) & \to \mathrm{H}_{p+q}(K^p, K^{p-1}) \end{array}$$

$$\begin{tikzcd} \mathrm{H}_{p+q}(K^{p-1}) \arrow[rr, "i_{p-1,q+1}"] & & \mathrm{H}_{p+q}(K^p) \arrow[ddl, "j_{p,q}"] \\ & \mathrm{H}_{p+q-1}(K^{p-1}) & \\ & \mathrm{H}_{p+q}(K^p, K^{p-1}) \arrow[u, "k_{p,q}"] \arrow[uul, bend left, "deg(-1)"] & \end{tikzcd}$$

6.2 The Spectral Sequence of a Filtered Complex

Here we construct a spectral sequence of filtered chain complexes C

$$\emptyset = C_{-1} \subset C_0 \subset C_1 \subset \cdots \subset C_k = C$$

They usually come from the chain complexes of filtered topological spaces, i.e. a topological space X with subspaces: $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_k = X$, where

$$C_p = C(X_p) = \{\cdots \to C_{p,q} \xrightarrow{\partial_q} C_{p,q-1} \to \dots\}$$

So we have nested chain sub-complexes of C with competible boundary operators.

$$\begin{array}{ccccccc} C_p & = & \{\cdots \to & C_{p,q} & \xrightarrow{\partial_q} & C_{p,q-1} & \to \dots\} \\ \cup & & & \cup & & \cup & \\ C_{p-1} & = & \{\cdots \to & C_{p-1,q} & \xrightarrow{\partial_q} & C_{p-1,q-1} & \to \dots\} \\ \cup & & & \cup & & \cup & \\ \vdots & & & \vdots & & \vdots & \end{array}$$

Definition 6.1. $\alpha \in C$ has *filtration index* p if $\alpha \in C_{p,q}$ but $\alpha \notin C_{p-1,q}$.

$$C_{p,q} = \{\text{elements of filtration index} \leq p\} \subset C_q$$

Definition 6.2. $E^0_{p,q} = C_{p,q}/C_{p-1,q} = C_q(X_p)/C_q(X_{p-1}) = C_q(X_p, X_{p-1})$

We have the projection

$$\begin{array}{ccc} C_{p,q} & \xrightarrow{proj} & E^0_{p,q} \\ \alpha & \dashrightarrow & \alpha_p \end{array}$$

So for $a, b \in C_{p,q}$ $a_p = b_p \iff a - b \in C_{p-1,q}$

Each row below gives homology of the corresponding complex

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{q+1} \arrow[r, "\partial_{q+1}"] \arrow[ddr, "d^r"] & C_q \arrow[r, "\partial_q"] & C_{q-1} \arrow[r] & \cdots \\
& \begin{array}{c}\cup\\ \vdots\\ \cup\end{array} & \begin{array}{c}\cup\\ \vdots\\ \cup\end{array} & \begin{array}{c}\cup\\ \vdots\\ \cup\end{array} & \\
\cdots \arrow[r] & C_{p+r,q+1} \arrow[r, "\partial_{q+1}"] & C_{p+r,q} \arrow[r, "\partial_q"] \arrow[ddr, "d^r"] & C_{p+r,q-1} \arrow[r] & \cdots \\
& \begin{array}{c}\cup\\ \vdots\\ \cup\end{array} & \begin{array}{c}\cup\\ \vdots\\ \cup\end{array} & \begin{array}{c}\cup\\ \vdots\\ \cup\end{array} & \\
\cdots \arrow[r] & C_{p,q+1} \arrow[r, "\partial_{q+1}"] & C_{p,q} \arrow[r, "\partial_q"] & C_{p,q-1} \arrow[r] & \cdots \\
& \begin{array}{c}\cup\\ \vdots\end{array} & \begin{array}{c}\cup\\ \vdots\end{array} & \begin{array}{c}\cup\\ \vdots\end{array} &
\end{tikzcd}
$$

Definition 6.3. *Define r^{th} cycles, and r^{th} boundaries as follows:*

$$Z^r_{p,q} = \{a_p \in E^0_{p,q} \mid \partial a \in C_{p-r,q-1}\} \tag{6.1}$$

$$B^r_{p,q} = \{a_p \in E^0_{p,q} \mid a = \partial b, b \in C_{p+r-1,q+1}\} \tag{6.2}$$

$$E^r_{p,q} = Z^r_{p,q}/B^r_{p,q}$$

From the definitions for large $r \gg 1$ we have the followings

$$Z^\infty_{p,q} = Z_q(C_p)/Z_q(C_{p-1})$$

$$B^\infty_{p,q} = B_q \cap C_{p,q}/B_q \cap C_{p-1,q}$$

Definition 6.4. $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$

Now construct thw chain complex $E^r = \{E^r_{p,q}, d^r_{p,q}\}$, $d^r_{p,q}\{a_p\} = \{(\partial a)_{p-r}\}$

$$
\begin{tikzcd}
\ldots \arrow[r] & E^r_{p+r,q+1} \arrow[r, "d^r_{p+r,q+1}"] \arrow[d, equal] & E^r_{p,q} \arrow[r, "d^r_{p,q}"] \arrow[d, equal] & E^r_{p-r,q-1} \arrow[r] \arrow[d, equal] & \ldots \\
& Z^r_{p+r,q+1}/B^r_{p+r,q+1} & Z^r_{p,q}/B^r_{p,q} & Z^r_{p-r,q-1}/B^r_{p-r,q-1} &
\end{tikzcd}
$$

$d^r_{p,q} \circ d^r_{p+r,q+1}\{a_{p+r}\} = d^r_{p,q}\{(\partial a)_p\} = \{(\partial\partial a)_{p-r}\} = 0$.

Theorem 6.5. *$H_q(E^r_{p,q}, d^r_{p,q}) \cong E^{r+1}_{p,q} = Z^{r+1}_{p,q}/B^{r+1}_{p,q}$, in short $H_*(E^r) = E^{r+1}$*

Proof. We first define a map $\phi : Ker(d^r_{p,q}) \to E^{r+1}_{p,q}$ as follows

$$\begin{aligned} & d^r_{p,q}\{a_p\} = \{(\partial a)_{p-r}\} = 0 \in E^r_{p-r,q-1} = Z^r_{p-r,q-1}/B^r_{p-r,q-1} \\ \Rightarrow\ & (\partial a)_{p-r} \in B^r_{p-r,q-1} \\ \Rightarrow\ & (\partial a)_{p-r} = (\partial b)_{p-r} \text{ for some } b \in C_{p-1,q} \implies (a-b)_p = a_p \\ \Rightarrow\ & (\partial a - \partial b)_{p-r} \in C_{p-r-1,q-1} \\ \Rightarrow\ & a - b \in Z^{r+1}_{p,q} \end{aligned}$$

Now let $a' := (a-b)_p$ and define $\phi\{a_p\} = \{a'\} \in E^{r+1}_{p,q}$

By construction ϕ is well defined (i.e. it doesn't depend on the choice of b)

Claim 1: ϕ is $1-1$, i.e. $Ker(\phi) = Im(d^r_{p+r,q+1})$

proof: Let $\{a_p\} \in Z^r_{p,q}$ with $\phi\{a_p\} = 0 \implies \{a_p\} \in B^{r+1}_{p,q}$
Hence $\exists\ b \in C_{p+r,q+1}$ with $(\partial b)_p = a_p$
Therefore $\{a_p\} = d^r_{p+r,q+1}(b_p)$

Claim 2: ϕ is onto

proof: Let $\{b_p\} \in E^{r+1}_{p,q} \implies b_p \in Z^{r+1}_{p,q} \implies \partial b \in C_{p-r-1,q}$
Hence $(\partial b)_{p-r} = 0 \implies d^r_{p,q}\{b_p\} = 0 \implies \{b_p\} \in ker(d^r_{p,q})$.
Therefore $\phi\{b_p\} = \{b_p\}$

□

For example, $0 = B^0_{p,q} \subset Z^0_{p,q} = E^0_{p,q} = C_{p,q}/C_{p-1,q}$, the boundary operators of $\{E^0_{p,q}, d^0_{p,q}\}$ are just the boundary operators of the relative chains

$$d^0_{p,q} = \partial_q = C_q(C_p, C_{p-1}) \to C_{q-1}(C_p, C_{p-1}) \implies$$

$$E^1_{p,q} = Z^1_{p,q}/B^1_{p,q} = Z_q(C_p, C_{p-1})/B_q(C_p, C_{p-1}) = H_q(C_p, C_{p-1})$$

Definition 6.6. ${}_{(p)}H_q(C) = Im(H_q(C_p) \to H_q(C))$ Hence

$$0 = {}_{(-1)}H_q(C) \subset {}_{(1)}H_q(C) \subset \cdots \subset {}_{(k)}H_q(C) = H_q(C)$$

Theorem 6.7. $E^{\infty}_{p,q} = Z^{\infty}_{p,q}/B^{\infty}_{p,q} = {}_{(p)}H_q(C)/{}_{(p-1)}H_q(C)$

Proof. By definitions we have

$$Z^{\infty}_{p,q} = Z_q(C_p)/Z_q(C_{p-1})$$

$$B^{\infty}_{p,q} = B_q \cap C_{p,q}/B_q \cap C_{p-1,q}$$

$${}_{(p)}H_q(C) = Z_q(C_p)/B_q \cap C_{p,q}$$

$${}_{(p-1)}H_q(C_{p-1}) = Z_q(C_p)/B_q \cap C_{p-1,q}$$

$\Phi : Z^{\infty}_{p,q} \to_p \mathrm{H}_q(C) \diagup {}_{p-1}\mathrm{H}_q(C)$ with $\Phi(x) = \pi g^{'} \beta^{'-1}(x)$, induces the desired isomorphism.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B_q(C) \cap C_{p-1,q} \arrow[r, "\alpha"] \arrow[d, "f^{''}"] & B_q(C) \cap C_{p,q} \arrow[r, "\beta"] \arrow[d, "f^{'}"] & B^{\infty}_{p,q} \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & Z_q(C_{p-1}) \arrow[r, "\alpha^{'}"] \arrow[d, "g^{''}"] & Z_q(C_p) \arrow[r, "\beta^{'}"] \arrow[d, "g^{'}"] & Z^{\infty}_{p,q} \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & {}_{p-1}\mathrm{H}_q(C) \arrow[r, "\alpha^{''}"] \arrow[d] & {}_{p}\mathrm{H}_q(C) \arrow[r, "\beta^{''}"] \arrow[d] \arrow[ddr, "\pi"] & E^{\infty}_{p,q} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & \\
 & & & {}_{p}\mathrm{H}_q(C) \diagup {}_{p-1}\mathrm{H}_q(C) \arrow[r] & 0
\end{tikzcd}$$

We can check well definedness by:

$$\Phi(x) = \pi g^{'}(\beta^{'-1}(x) + \alpha^{'}(z)) = \pi g^{'}(\beta^{'-1}(x)) + \alpha^{''}(g^{''}(z)) = \pi g^{'}(\beta^{'-1}(x))$$

$\square$

6.2.1 Change of notation:

Now we convert to a more commonly used convention of the litrature, by changing our indices $q \mapsto p+q$. The reason why we start with one notation then change to another one is, we feel first notation gives more intuitive understanding of spectral sequences. Now we change the notation $E = \{E^r_{p,q}\} \mapsto \mathbb{E} = \{\mathbb{E}^r_{p,q}\}$, where $\mathbb{E}^r_{p,q} := E^r_{p,q+p}$ and so $E^r_{p,q} = \mathbb{E}^r_{p,q-p}$, then the map $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q-1}$ becomes

$$d^r_{p,q} : \mathbb{E}^r_{p,q} \to \mathbb{E}^r_{p-r,q+r-1}$$

In this change ${}_{(p)}\mathrm{H}_q(C)$ has to be replaced by ${}_{(p)}\mathrm{H}_{p+q}(C) = \mathrm{im}(\mathrm{H}_{p+q}(C_p) \to \mathrm{H}_{p+q}(C))$. From now on we will use this new convention, and just use the notation $E^r_{p,q}$ for $\mathbb{E}^r_{p,q}$.

Also by dualizing filtered chain complexes we get filtered cochain complexes, and by similar definitions we end up by a spectral sequence of in cohomology $\{E^{p,q}_r, d^{p,q}_r\}$

$$d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

satisfying $H^*(E_r) = E_{r+1}$, and obtain subgroups ${}_{(p)}H^*(C) = \ker(H^*(C) \to H^*(C^p))$ inducing the filtration $0 = {}_{(-1)}H^{p+q}(C) \subset {}_{(0)}H^{p+q}(C) \subset \cdots \subset {}_{(-1)}H^{p+q}(C) = H^{p+q}(C)$, where ${}_{(p)}H^*(C) = \ker(H^*(C) \to H^*(C^p))$.

Recall, a topological space X with subspaces $0 \subset X_{-1} \subset X_0 \subset \cdots \subset X_k = X$ induces filtration of chain complexes: $C_{-1} \subset C_0 \subset \cdots \subset C_k = C(X)$, with $C_i = C_*(X_i)$. which then we can form the corresponding (homology or cohomology) spectral sequences.

Theorem 6.8 (Leray-Homology version)**.** *If the space X is filtred by subspaces $0 \subset X_{-1} \subset X_0 \subset \cdots \subset X_k = X$ then $\exists$ groups $E^r_{p,q}$ $(= 0$ when $p < 0$ or $p > k)$ and homomorphisms $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ with $d^r_{p,q} \circ d^r_{p+r,q-r+1} = 0$ such that*

1. $E^{r+1}_{p,q} = ker(d^r_{p,q})/im(d^r_{p+r,q-r+1})$

2. $E^0_{p,q} = C_{p+q}(X_p, X_{p-1})$

3. $E^\infty_{p,q} =_{(p)} H_{p+q}(X)/_{(p-1)}H_{p+q}(X)$

where ${}_{(p)}H_{p+q}(X) = im(H_{p+q}(X_p) \to H_{p+q}(X))$ *giving the filitration*
$0 = {}_{(-1)}H_{p+q}(X) \subset {}_{(0)}H_{p+q}(X) \subset \cdots \subset {}_{(k)}H_{p+q}(X) = H_{p+q}(X)$.

Theorem 6.9 (Leray-Cohomology version)**.** *If the space* X *has filtration* $0 \subset X_{-1} \subset X_0 \subset \cdots \subset X_k = X$ *then* $\exists$ *groups* $E_r^{p,q}$ $(= 0$ *when* $p < 0$ *or* $p > k)$ *and homomorphisms* $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ *with* $d_r^{p,q} \circ d_r^{p+r,q-r+1} = 0$ *and*

1. $E_{r+1}^{p,q} = ker(d_r^{p,q})/im(d_r^{p-q,q+r-1})$
2. $E_0^{p,q} = C^{p+q}(X_p, X_{p-1})$
3. $E_\infty^{p,q} = {}_{(p-1)} H^{p+q}(X)/{}_{(p)}H^{p+q}(X)$

where ${}_{(p)}H^{p+q}(X) = ker(H^{p+q}(X) \to H^{p+q}(X_p))$ *giving the filitration*
$0 = {}_{(k)}H^{p+q}(X) \subset {}_{(0)}H^{p+q}(X) \subset \cdots \subset {}_{(-1)}H^{p+q}(X) = H^{p+q}(X)$.

Example 6.1. *Let* $X = CW-complex$ *with* $dim(X) = k$, $X^i =$ *cells of dim* $\leq i$
$\implies C_{-1} \subset C_0 \subset \cdots \subset C_k = C(X)$ *where* $C_i = C_*(X^i)$

$$E^1_{p,q+p} = H_{p+q}(X^p, X^{p-1}) = \begin{cases} 0 & if \quad q \neq 0 \\ \mathcal{C}_p(X) & if \quad q = 0 \end{cases}$$

Hence $d^1_{p,q} : E^1_{p,q} = 0 \to E^1_{p-1,q} = 0$ *is the zero map, when* $q \neq 0$, *and the map*

$$\begin{array}{cccc} d^1_{p,0}: & E^1_{p,0} & \longrightarrow & E^1_{p-1,0} \\ & \shortparallel & & \shortparallel \\ & \mathcal{C}_p(X) & \xrightarrow{\partial} & C_{p-1}(X) \\ & \shortparallel & & \shortparallel \\ & H_p(X^p, X^{p-1}) & & H_{p-1}(X^{p-1}, X^{p-2}) \end{array}$$

comes from the long exact sequence of triple $X^{p-2} \subset X^{p-1} \subset X^p$. *Also we have* $E^2_{p,q} = 0$ *when* $q \neq 0$ *and so* $E^2_{p,0} = E^3_{p,0} = \cdots = H_p(X)$

6.3 Spectral Sequence of Fibration

Let $F \to E \to B$ be a fibrations of CW-complexes. Let $\emptyset = B^{-1} \subset B^1 \subset \cdots \subset B^k = B$ be the filtration by cells. This gives the filtration $\emptyset = E_{-1} \subset E_1 \subset \cdots \subset E_k = E$, where $E_i = p^{-1}(B^i)$. Given $x_0, x_1 \in B$, and a path γ connecting them induces a homotopy equivalence $\gamma^\sharp_{x_0,x_1} : F_{x_0} \xrightarrow{\simeq} F_{x_1}$ and homotopic loops induce homotopic maps. After fixing $F = F_{x_0}$, we get an action $\pi_1(B)$ on F, which makes $H_*(F)$ a $\mathbb{Z}[\pi_1(B)]$-module. Then we can define homology with coefficients by $H_p(B, H_q(F)) = H_p(\tilde{C})$, where $\tilde{C} = C_*(\tilde{B}) \otimes_{\mathbb{Z}[\pi_1(B)]} H_q(F)$ by using the action. If the action of $\pi_1(B)$ on F trivial call the fibration simple. Here for smplicity we will assume all the fibrations are simple. Writing $B^p = B^{p-1} + \sum e^p_i$, we identify $E^1_{p,q}$ by the p-chains of B coefficients in $H_q(F)$:

$$\begin{aligned} E^1_{p,q} &= \mathrm{H}_{p+q}(E^p/E^{p-1}) = H_{p+q}(\sqcup_i \pi^{-1}(e^p_i)/\pi^{-1}(\partial e^p_i)) && (6.3)\\ &= H_{p+q}(\sqcup D^p_i \times F/S^{p-1}_i \times F) \cong H_p(B^p, B^{p-1}) \otimes H_q(F) && (6.4)\\ &= \mathcal{C}_p(B) \otimes H_q(F) = \mathcal{C}_p(B, H_q(F)) && (6.5) \end{aligned}$$

$d^1_{p,q} : C_p(B, \mathrm{H}_q(F)) \to C_{p-1}(B, \mathrm{H}_q(F))$ is the usual boundary map, so $E^2_{p,q} = \mathrm{H}_p(B, \mathrm{H}_q(F))$

Theorem 6.10 (Serre-Homology version)**.** *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a simple fibration. Then $\exists$ a spectral sequence $E^r = \left\{E^r_{p,q}, d^r\right\}$ with $E^2_{p,q} = H_p(B, H_q(F))$, and a flitration*

$$F_{0,n} \subset F_{1,n-1} \subset \cdots \subset F_{n,0} = H_n(E) \quad \textit{such that} \;\; F_{p,q}/F_{p-1,q+1} = E^\infty_{p,q}$$

Induced maps π_* and i_* can be computed from the diagrams (Section 6.3.1), $r >> 1$:

$$\begin{array}{ccccc} H_n(E) & & \xrightarrow{\pi_*} & & H_n(B) \\ \shortparallel & & & & \shortparallel \\ F_{n,0} \longrightarrow E^\infty_{n,0} & = & E^r_{n,0} & \subset & E^2_{n,0} \end{array} \qquad \begin{array}{ccccc} H_n(F) & & \xrightarrow{i_*} & & H_n(E) \\ \shortparallel & & & & \cup \\ E^2_{n,0} \longrightarrow E^r_{0,n} & & = & & E^\infty_{0,n} = F_{0,n} \end{array}$$

Theorem 6.11 (Serre-Cohomology version). *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a simple fibration. Then $\exists$ a spectral sequence $E_r = \{E_r^{p,q}, d_r\}$ with $E_2^{p,q} = H^p(B, H^q(F))$ and a flitration*

$$H^n(E) = F^{0,n} \supset F^{1,n-1} \supset \cdots \supset F^{n,0} \supset 0 \quad \text{such that } F^{p,q}/F^{p+1,q-1} = E_\infty^{p,q}$$

and the induced maps π^* and i^* can be computed from the diagrams (Section 6.3.1):

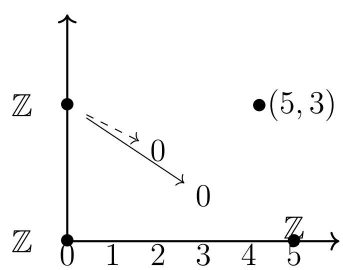
$$\begin{array}{ccccc} H^n(E) & & \xrightarrow{i^*} & & H^n(F) \\ \shortparallel & & & & \shortparallel \\ F^{0,n} & \longrightarrow E_\infty^{0,n} & = & E_r^{0,n} & \subset E_2^{0,n} \end{array} \qquad \begin{array}{ccccc} H^n(B) & & \xrightarrow{\pi^*} & & H^n(E) \\ \shortparallel & & & & \cup \\ E_2^{n,0} & \longrightarrow E_r^{n,0} & = & E_\infty^{n,0} & = F^{n,0} \end{array}$$

($r >> 1$). Furthermore, the maps $d_r^{p,q}$ are anti-derivations, i.e. the following holds

$$d_r(a, b) = d_r(a)b + (-1)^{p+q} a.d_r(b) \tag{6.6}$$

and d_r are induced by the cup product map $E_2^{p,q} \otimes E_2^{p',q'} \to E_2^{p+p',q+q'}$.

Example 6.2. *Take the fibration $SU(n-1) \to SU(n) \to S^{2n-1}$. We can compute $H_*(SU(3))$ from the the speial case of this fibration: $S^3 \to SU(3) \to S^5$*



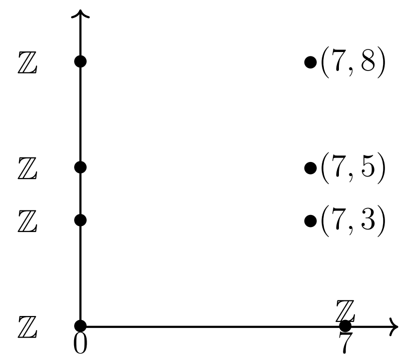
$$E_{p,q}^2 = H_p(S^5, H_q(S^3)) = 0 \text{ when } p \neq 0 \text{ or } 5, \text{ and so}$$

$$0 = E_{7,1}^2 \xrightarrow{d^2} E_{5,0}^2 \xrightarrow{d^2} E_{3,-1}^2 = 0$$

Hence $d^2 = d^3 = \cdots = 0 \implies E^2 = E^\infty \implies H_0 = H_3 = H_5 = H_8 = \mathbb{Z}$, *which implies*

$$H_*(SU(3)) = H_8(S^3 \times S^5)$$

Similarly we can compute $H_*(SU(4))$ from the fibration $SU(3) \to SU(4) \to S^7$



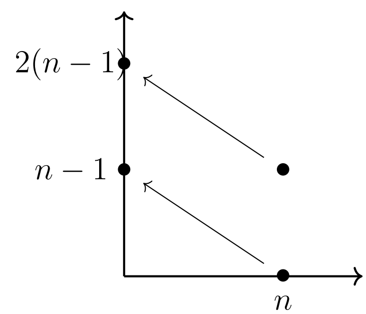
The nonzero terms of $E^2_{p,q}$ are indicated in the E^2 page above $\implies$ $E_2 = E^3 = \cdots = E^\infty$ so $H_0 = H_5 = H_7 = H_8 = H_{10} = H_{12} = H_{15} \cong \mathbb{Z}$, and $H_q(SU(4)) = H_q(S^3 \times S^5 \times S^7, \mathbb{Z})$.

Example 6.3. *Computing homology of the loop space $H_*(\Omega S^n)$ from the fibration*

$$\Omega S^n \to \mathcal{P}S^n \to S^n \quad (\mathcal{P}S^n \ \text{ is contractible})$$

$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ are all zero, except the following case (k=1,2,...)

$$d^n : E^n_{n,k(n-1)} \xrightarrow{\cong} E^n_{0,(k+1)(n-1)}$$



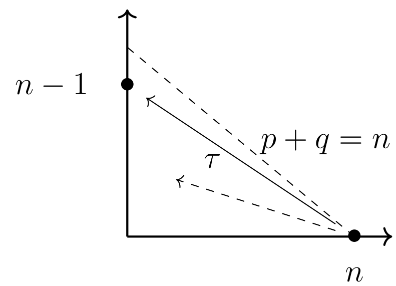
$$\textit{Hence } \ H_q(\Omega S^n) = \begin{cases} \mathbb{Z} & \textit{if} \quad q = k(n-1) \\ 0 & \quad \textit{otherwise} \end{cases}$$

Theorem 6.12 (Serre exact sequence). *Given filtration* $F \xrightarrow{i} E \xrightarrow{\pi} B$ *where* B *is* $(p-1)$ *connected and* F *is* $(q-1)$ *connected then we have the following exact sequences.*

(a) $H_{p+q-1}(F) \xrightarrow{i_*} H_{p+q-1}(E) \xrightarrow{\pi_*} H_{p+q-1}(B) \xrightarrow{\tau} H_{p+q-2}(F) \to \cdots \to H_1(F) \to 0$

(b) $0 \to H^1(F) \to \cdots \to H^i(B) \xrightarrow{\pi^*} H^i(E) \xrightarrow{i^*} H^i(F) \xrightarrow{\tau} H^{i+1}(B) \to \cdots \to H^{p+q-1}(F)$

Proof. (a) $E^2_{i,j} = 0$ when $0 < i < p$ or $0 < j < q \implies E^\infty_{i,j} = 0$ when $0 < i + j < p + q$



Now consider the filtration $\mathrm{H}_n(E) = F_{n,0} \supset F_{n-1,1} \supset \cdots \supset F_{0,n}$. Then when $n < p+q$ this filtration reduces to $\mathrm{H}_n(E) = F_{n,0} \supset F_{0,n} = E^\infty_{n,0}$ implying the exact sequence:

$$0 \to E^\infty_{0,n} \xrightarrow{\alpha_n} \mathrm{H}_n(E) \xrightarrow{\beta_n} E^\infty_{n,0} \to 0$$

By this and definitions we get the following exact, sequence, whose iteration gives (a)

$$\begin{tikzcd}
0 \arrow[r] & E^\infty_{n,0} \arrow[r] & E^n_{n,0} \arrow[r, "d^n"] \arrow[d, equal] & E^n_{0,n-1} \arrow[r] \arrow[d, equal] & E^\infty_{0,n-1} \arrow[r] \arrow[d, equal] & 0 \\
 & & E^2_{n,0} \arrow[d, equal] & E^2_{0,n-1} \arrow[d, equal] & F_{0,n-1} \arrow[d] & \\
im(\alpha_n) \arrow[r, hook] & \mathrm{H}_n(E) \arrow[uu] \arrow[r, dashed, "\pi_*"'] \arrow[ur, dashed, "\pi_*"] & \mathrm{H}_n(B) \arrow[r, dashed, "\tau"'] & \mathrm{H}_{n-1}(F) \arrow[r, dashed, "i_*"'] \arrow[ur, dashed, "i_*"] & \mathrm{H}_{n-1}(E) \arrow[r, two heads] & ker(\beta_{n-1})
\end{tikzcd}$$

Proof of (b) is similar, when $n < p+q$ the flitration $\mathrm{H}^n(E) = F^{0,n} \supset F^{1,n-1} \supset \cdots \supset F^{n,0}$ reduces to $\mathrm{H}^n(E) = F^{0,n} \supset F^{n,0} = E^{n,0}_\infty$ implying the exact sequence:

$$0 \to E^{n,0}_\infty \to \mathrm{H}^n(E) \to E^{0,n}_\infty \to 0$$

As above, by splicing this with the following exact sequence gives the proof.

$$\begin{tikzcd}
0 \arrow[r] & E_\infty^{0,n-1} \arrow[r] & E_n^{0,n-1} \arrow[r, "\tau"] \arrow[d, equal] & E_n^{n,0} \arrow[r] \arrow[d, equal] & E_\infty^{n,0} \arrow[r] \arrow[d, equal] & 0 \\
 & & E_2^{0,n-1} \arrow[d, equal] & E_2^{n,0} \arrow[d, equal] & F^{n,0} \arrow[d] & \\
 & \mathrm{H}^n(E) \arrow[uu] \arrow[r, dashed, "i^*"'] \arrow[ur, dashed] & \mathrm{H}^{n-1}(F) \arrow[r, dashed, "\tau"'] & \mathrm{H}^n(B) \arrow[r, dashed, "\pi^*"'] \arrow[ur, dashed] & \mathrm{H}^n(E) &
\end{tikzcd}$$

□

Theorem 6.13 (Freudental)**.** *If X is $(n-1)-$connected, and $i < 2n-1$ then*

$$\pi_i(X) \cong \pi_{i+1}(\Sigma X)$$

Proof. Consider the path space fibration $\Omega X \to \mathcal{P}X \to X.\mathcal{P}X$ is contactable, and since X is $(n-1)$ connected ΩX is $(n-2)$ connected. Then the Serre exact sequence gives

$$\mathrm{H}_{i-1}(\Omega X) \xrightarrow[\cong]{\tau} \mathrm{H}_i(X) \quad \forall i \ \leq 2n-2 \tag{6.7}$$

Recall the definition of reduced suspension $\Sigma X = (X \times I)/(X \times 0) \cup (X \times 1) \cup (x_0 \times I)$. Denote elements of ΣX by $[x, t]$, then define the following maps:

$$\begin{array}{ccccc}
\Sigma\Omega X & \xrightarrow{\pi^X} & X & \xrightarrow{i^X} & \Omega\Sigma X \\
 & & x & \longrightarrow & (t \mapsto [x, t]) \\
\left[\phi, t\right] & \longrightarrow & \phi(x) & &
\end{array}$$

From 6.7 it follows $\forall i \leq 2n-2$ we have isomorphisms

$$\begin{tikzcd}
\pi_*^X : \mathrm{H}_i(X) \arrow[r] \arrow[rr, bend right, "\cong"'] & \mathrm{H}_i(\Sigma\Omega X) \arrow[r, phantom, "\cong"] & \mathrm{H}_{i-1}(\Omega X)
\end{tikzcd}$$

The composition of the following maps is identity

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma\, i^X"] \arrow[rr, bend right, "id"'] & \Sigma\Omega(\Sigma X) \arrow[r, "\pi^{\Sigma X}"] & \Sigma X \end{tikzcd}$$

$$\begin{array}{ccccc} [x,s] & \longmapsto & [\, i_X(x), s\,] & \longmapsto & [x,s] \\ & & \shortparallel & & \\ & & t \mapsto (x,t) & & \end{array}$$

Hence composition in homology is identity. By (6.7) the second map is an isomorphism when $i \leq 2(n+1) - 2$, so the first map $(\Sigma\, i^X)_*$ is an isomorphism when $i \leq 2n$.

$$\begin{array}{ccccc} \mathrm{H}_i(\Sigma X) & \xrightarrow{(\Sigma\, i^X)_*} & \mathrm{H}_i(\Sigma\Omega\Sigma X) & \underset{\cong}{\xrightarrow{(\pi^{\Sigma X})_*}} & \mathrm{H}_i(\Sigma X) \\ \wr\| & & \wr\| & \uparrow & \\ \mathrm{H}_{i-1}(X) & \xrightarrow{(i^X)_*} & \mathrm{H}_{i-1}(\Omega\Sigma X) & (\ i \leq 2n\ ) & \end{array}$$

Therefore the middle map of the long exact sequence of the pair $X \hookrightarrow \Omega\Sigma X$ below is an isomorphism when $i \leq 2n - 1$

$$\mathrm{H}_{i+1}(\Omega\Sigma X, X) \to \mathrm{H}_i(X) \xrightarrow{i_X} \mathrm{H}_i(\Omega\Sigma X) \to \mathrm{H}_i(\Omega\Sigma X, X)$$

$$\text{Therefore } \ \mathrm{H}_i(\Omega\Sigma X, X) = 0 \ \text{ for } i \leq 2n-1$$
$$\text{Hence } \ \pi_i(\Omega\Sigma X, X) = 0 \ \text{ for } i \leq 2n-1 \quad \Longrightarrow$$
$$\pi_i(X) \cong \pi_i(\Omega\Sigma X) \cong \pi_{i+1}(\Sigma X) \ \text{ for } i < 2n-1$$

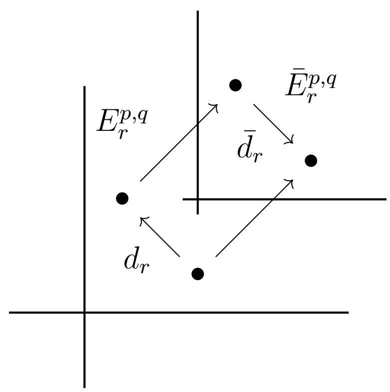
$\square$

6.3.1 Spectral Sequence Comparison

Let (E, B, F, π), $(\bar{E}, \bar{B}, \bar{F}, \bar{\pi})$ be two fibrations of CW-complexes with with commuting maps between them:

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "\pi"] & \bar{E} \arrow[d, "\bar{\pi}"] \\
B \arrow[r] & \bar{B}
\end{tikzcd}$$

Then we get a map $E^2_{p,q} \longrightarrow \bar{E}^2_{p,q}$ which commutes with d_2, which is induced from the natural map $\mathrm{H}_p(B, \mathrm{H}_q(F)) \to \mathrm{H}_p(\bar{B}, \mathrm{H}_q(\bar{F})) \implies$ For $r \geq 2$ and $\forall p, q$ we get maps $E^r_{p,q} \to \bar{E}^r_{p,q}$ commuting with d_r



Corollary 6.14. *Up to isomorphism $\{E_r^{p,q}, d^r\}$ is independent of CW structure.*

Given a fibration $F \xrightarrow{i} E \xrightarrow{\pi} B$, we can compute induced maps i_* and π_* (or i^*, π^*) from its spectral sequence by using the spectral sequence comparison. For example, comparing it to the spectral sequence $\{'E^r_{p,q}, 'd_r\}$ of the trivial fibration $F \to F \to *$ (where $*$ is a point), we get $0 = {}'E^2_{p,q} \to E^2_{p,q}$ if $p > 0$, and the commuting diagrams:

$$
\begin{tikzcd}
F \arrow[r, "="] \arrow[d, "="'] & F \arrow[d, "="'] \\
F \arrow[r, "i"'] \arrow[d] & E \arrow[d] \\
* \arrow[r] & B
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
{}'E^2_{0,q} \arrow[r, "="] \arrow[d, "="'] & E^2_{0,q} \arrow[d] \arrow[ddd, bend left] & = \mathrm{H}_q(F) \\
{}'E^r_{0,q} \arrow[d, equal] & E^r_{0,q} \arrow[d, equal] & \\
{}'E^\infty_{0,q} \arrow[d, equal] & E^\infty_{0,q} \arrow[d, hook] & \\
\mathrm{H}_q(F) \arrow[r, "i_*"] & \mathrm{H}_q(E) &
\end{tikzcd}
$$

If we compare the fibration to the trivial fibration $* \to B \to B$ we get:

$$
\begin{tikzcd}
F \arrow[r] \arrow[d] & * \arrow[d] \\
E \arrow[r] \arrow[d] & B \arrow[d, "j"] \\
B \arrow[r, "="] & B
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{array}{ccccccccc}
\mathrm{H}_q(E) & = & F_{q,0} & \to & E^\infty_{q,0} & \overset{r>>1}{=} & E^r_{q,0} & \hookrightarrow & \mathrm{H}_q(B) \\
\downarrow \pi_* & & & & & & & & \parallel \\
\mathrm{H}_q(B) & = & F_{q,0} & = & E^\infty_{q,0} & = & E^r_{q,0} & = & \mathrm{H}_q(B)
\end{array}
$$

6.3.2 Transgression

Given a fibration $\pi : (E, F) \to (B, *)$ the transgression maps τ_* and τ^* are first defined (up to indeterminacies) as follows:

$$\tau_*(\pi_*(\beta)) = \partial\beta \quad \Longleftrightarrow \quad \tau_*(\alpha) = \partial(\pi^{-1}(\alpha)) \tag{6.8}$$

$$\tau^*(\alpha) = \beta \quad \text{if} \quad \pi^*(\beta) = \delta(\alpha) \tag{6.9}$$

$$
\begin{tikzcd}
\mathrm{H}_q(B) = \mathrm{H}_q(B, *) \arrow[r, dashed, bend right, "?"'] \arrow[rr, bend right, "\tau_*"'] & \mathrm{H}_q(E, F) \arrow[l, "\pi_*"'] \arrow[r, "\partial"] & \mathrm{H}_{q-1}(F)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{H}^q(F) \arrow[r, "\delta"] \arrow[rr, bend right, "\tau^*"'] & \mathrm{H}^{q+1}(E, F) \arrow[r, dashed, bend right, "?"'] & \mathrm{H}^{q+1}(B, *) = \mathrm{H}^{q+1}(B) \arrow[l, "\pi^*"']
\end{tikzcd}
$$

To make them well defined we have define them in the following quotient groups:

$$\begin{array}{rll}\tau_* : & \mathrm{im}(\pi_*) & \to \mathrm{H}_{q-1}(F)\big/\partial(\ker\pi_*) \\ \tau^* : & \delta^{-1}(\mathrm{im}(\pi^*)) & \to \mathrm{H}^{q+1}(B)\big/\ker(\pi^*)\end{array}$$

Lemma 6.15.

$$\begin{array}{lllll}(a) & \tau_* = d^n & : & E^n_{n,0} & \to E^n_{0,n-1} \\ (b) & \tau_* = d_{n+1} & : & E^{0,n}_{n+1} & \to E^{n+1,0}_{n+1}\end{array}$$

Proof. (Rough sketch) Recall in the old notation (Section 6.2.1) we defined

$$Z^r_{p,q} = \{\alpha \in C_q(E^p, E^{p-1}) \mid \partial\alpha \in C_{q-1}(E^{p-r})\}$$

So in the new notation of $Z^n_{n,0}$ corresponds to $Z^n_{n,n}$ in the old notation, so

$$Z^n_{n,0} = \{\alpha \in C_n(E^n, E^{n-1}) \mid \partial\alpha \in C_{n-1}(F)\}$$

where $E^0 = p^{-1}(x_0)$ is the fiber F, also $d^n(\alpha) = \partial\alpha$. So the subgroup $E^n_{n,0} \subset E^2_{n,0} = H_n(B)$, consists of classes which are represented by chains in $C_n(E)$ whose boundaries lie in F. Then the description of π_* in Section 6.3.1 implies (a). Proof of (b) is similar. □

We say a class in $H_n(B)$ (or in $H^n(F)$) is trangressive if $\tau_*(x)$ (or $\tau^*(x)$) is defined.

Proposition 6.16. *If x transgressive $\implies Sq^i(x)$ is transgressive*

Proof. $y \in \tau(x) \implies Sq^i(y) \in \tau(Sq^i(x))$

$$\begin{tikzcd} \mathrm{H}^n(F) \arrow[r, "\delta"] \arrow[rr, bend right, "\tau"'] & \mathrm{H}^{n+1}(E,F) \arrow[r, dashed, bend right] & \mathrm{H}^{n+1}(B,*) \arrow[l, "\pi^*"'] \end{tikzcd}$$

If $y \in \tau(x) \implies \pi^*(y) = \delta x \implies \begin{array}{ccc} Sq^i(\pi^*(y)) & = & Sq^i(\delta x) \\ \shortparallel & & \shortparallel \\ \pi^*(Sq^i(y)) & & \delta(Sq^i(x)) \end{array} \implies \tau(Sq^i(x)) = Sq^i(y)$

□

6.3.3 Applications of spectral sequence comparison

Spectral sequences as purely algebraic objects. The spectral sequences $E = \{E_r^{p,q}, d_r^{p,q}\}$ we obtained from fibrations has the property $E_r^{p,q} = 0$ when $p < 0$ or $q < 0$ (i.e they are first quadrant spectral sequences), and $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$ (at least when we used field cefficent cohomology groups). Call such a spectral sequence *standard spectral sequence.*

Theorem 6.17. *Let $E = \{E_r^{p,q},\}$ and $\bar{E} = \{\bar{E}_r^{p,q}\}$ be standard specral sequences, and let $f : E \to \bar{E}$ be a homomorphism, then*

$$\left.\begin{array}{ll} (a) & f_2^{p,q} : E_2^{0,q} \xrightarrow{\cong} \bar{E}_2^{0,q} \ \ \forall q \\ (b) & E_\infty^{p,q} = \bar{E}_\infty^{p,q} = 0, \ \forall (p,q) \neq (0,0) \end{array}\right\} \Rightarrow f_r^{p,q} \text{ is isomorphism } \forall r \geq 2, \ \forall (p,q)$$

Proof. Clearly if $f_2^{p,0}$ is isomorphism for all p and (a) implies $f_2^{p,q}$ is isomorphism $\forall (p,q)$, and hence $f_r^{p,q}$ is $\cong$ for all $r \geq 2$. So we only need to show $f_2^{p,0}$ is isomorphism $\forall p$.

We proceed by induction on p. Assume $f_2^{s,0} \cong \forall s < p$ (so $f_2^{s,q}$ and hence $f_r^{s,q}$ is $\cong$) Also note that for $r > p$ and $r \geq 2$ we have $E_r^{p,0} = E_\infty^{p,0} = 0 = \bar{E}_\infty^{p,0} = \bar{E}_r^{p,0}$.

$$0 = E_r^{p-r,r-1} \xrightarrow{d_r^{p-r,r-1}} E_r^{p,0} \xrightarrow{d_r^{p,0}} E_r^{p+r,-r+1} = 0$$

Hence in this range $f_r^{p,o}$ is $\cong$. Now we do downward induction on r, i.e. show that $f_{r+1}^{p,0}$ is an isomorphism $\Rightarrow f_r^{p,0}$ is an isomorphism . We have a commuting diagram

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{im}(d_r^{p-r,r-1}) & \to & E_r^{p,0} & \to & E_{r+1}^{p,0} & \to & 0 \\ & & \downarrow f_r^{p,0} & & \downarrow f_r^{p,0} & & \cong \downarrow f_{r+1}^{p,0} & & \\ 0 & \to & \operatorname{im}(\bar{d}_r^{p-r,r-1}) & \to & \bar{E}_r^{p,0} & \to & \bar{E}_{r+1}^{p,0} & \to & 0 \end{array}$$

From this diagram we see that we have a mapping $f_r^{p,0} : \operatorname{im}(d_r^{p-r,r-1}) \to im(\bar{d}_r^{p-r,r-1})$, which in fact must be an isomorphism by induction. Then by appying $5-$Lemma to this diagram we conclude that $f_r^{p,0}$ is an isomorphism.

□

Definition 6.18. We say a graded ring R is generated by simple set of generators $\{a_1, a_2, a_3 \dots\}$ if the monomials form $\{a_{i_1}a_{i_2}...a_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$ an additive basis for R. For example, $\{x_i^j\}$ is a simple set of generators for $R = \mathbb{Z}_2[x_1, x_2, \dots]$

Theorem 6.19 (Borel)**.** *Given a fibration $F \to E \to B$ where E is cyclic. Suppose $H^*(F;\mathbb{Z}_2)$ as a ring generated by a simple set of transgressive generators $\{x_\alpha\}$, then $H^*(B,\mathbb{Z}_2)$ is the polynomial ring generated by $\{\tau(x_\alpha)\}$.*

Proof. Let E be the given spectral sequence of this fibration, we will algebraically construct a spectral sequence $\bar{E}$ and a map $f : \bar{E} \to E$ and then apply comparison theorem.

Let $\Lambda = \wedge[\{z_\alpha\}]$ be an exterior algebra with $deg(z_\alpha) = deg(x_\alpha)$, and $P = \mathbb{Z}_2[\{y_\alpha\}]$ be a polynomial algebra with $deg(y_\alpha) = deg(x_\alpha) + 1$. Let $C = P \otimes \Lambda$, with (degree 1) differential $\delta : C \to C$ defined by, $\delta(y \otimes z_\alpha) = yy_\alpha \otimes 1$ and $\delta(y \otimes 1) = 0$. C is filtered by

$$C^p = \oplus_{i \geq p} P^i \otimes \wedge$$

This gives a spectral sequence $\bar{E}$ wih $\bar{E}_2^{p,q} = P^p \otimes \Lambda^q$ and $\bar{E}_\infty^{p,q} = 0 \ \forall (p,q) \neq (0,0)$

Define $f : \bar{E} \to E$ by defining the following ring homomorphisms:

$$\begin{aligned} f_2^{*,0} &: P \longrightarrow \mathrm{H}^*(B, \mathbb{Z}_2) \qquad && y_\alpha \mapsto \tau(x_\alpha) \\ f_2^{0,*} &: \wedge \longrightarrow \mathrm{H}^*(F, \mathbb{Z}_2) && z_{i_1} \dots z_{i_r} \mapsto x_{i_1} \dots x_{i_r} \end{aligned}$$

and let $f_2^{p,q} = f_2^{p,0} \otimes f_2^{0,q}$. Inductively after defining f_r define f_{r+1} by requiring that f_{r+1} be induced by f_r. This is possible since $f_r \bar{d}_r = d_r f_r$ (check this by using 6.6 and ising the fact that x_α is transgressive). Now apply the Theorem 6.17 and get a ring isomorphism

$$f_2^{*,0} : P \to \mathrm{H}^*(B; \mathbb{Z}_2)$$ □

Definition 6.20. A multi-index $I = (i_1, i_2, \dots i_r) \in \mathbb{Z}_+^r$ is called *admissable* if it satisfies $i \geq 2i_2, \ i_2 \geq 2i_3, \dots i_{r-1} \geq 2i_r$. Let $d(I) = i_1 + i_2 \cdots + i_r$, and define the *excess* of I to be

$$e(I) = (i_1 - 2i_2) + (i_2 - 2i_3) + \cdots + (i_{r-1} - 2i_r) + i_r = 2i_1 - d(I)$$

Corollary 6.21. *$H^*(K(\mathbb{Z}_2,q),\mathbb{Z}_2) = \mathbb{Z}_2[Sq^I(i_q)]$, where I runs through admissible sequences with $e(I) < q$.*

Proof. We prove this by induction on q, by using the path space fibration with contractible total space

$$K(\mathbb{Z}_2,q) \to P \to K(\mathbb{Z}_2,q+1) \tag{6.10}$$

By induction $\mathrm{H}^*(K(\mathbb{Z}_2,q),\mathbb{Z}_2) = \mathbb{Z}_2[z_j]$, where $z_j = Sq^I(i_q)$ are classes of dimension $a_j = d(I)+q$ for some admissable I with $e(I) < q$. In the fibration 6.10 the transgression maps fundamental classes to each other. $\tau(i_q) = i_{q+1}$. then since

$$\begin{array}{rcl} (z_j)^2 & = & Sq^{a_j}(z_j) \\ (z_j)^{2^2} & = & Sq^{2a_j}Sq^{a_j}(z_j) \\ \vdots & & \\ (z_j)^{2^r} & = & Sq^{2^{r-1}a_j}Sq^{2^{r-2}a_j}\dots Sq^{a_j}(z_j) = Sq^J(z_j) \end{array}$$

where $J = (2^{r-1}a_j, 2^{r-2}a_j, \dots, 2a_j, a_j)$. So by 6.16 $\{z_j^{2^r}\}$ is a simple system of transgrassive generators of $\mathrm{H}^*(F;\mathbb{Z}_2)$. So by Theorem 6.19 $\mathrm{H}^*(B;\mathbb{Z}_2)$ is a polynomial algebra:

$$\mathrm{H}^*(B;\mathbb{Z}_2) = \mathbb{Z}_2[\{\tau Sq^J z_j\}] = \mathbb{Z}_2[\{Sq^J\tau(z^j)\}] = \mathbb{Z}_2[\{Sq^JSq^I(i_{q+1})\}]$$

The following Lemma finishes the proof □

Lemma 6.22. *When I runs through admissible sequences with $e(I) < q \Rightarrow JI$ runs through all admissible sequences with $e(I) < q+1$.*

Proof. We construct bijection by first defining $\phi(I) = J.I$ (clearly JI is admissible), then by constructing its inverse ψ.

$$\left\{ I \;\middle|\; \begin{array}{c} \text{I admissible} \\ e(I) < q \end{array} \right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{ J \;\middle|\; \begin{array}{c} \text{J admissible} \\ e(J) < q+1 \end{array} \right\}$$

This is left as an exercise. □

Chapter 7

Postnikov and Whitehead towers

Let us first review some basic facts:

- Let $X \xrightarrow{i} X' = X \cup_\alpha e^{n+1}$. Then

  1. $\pi_k(X) \xrightarrow[\cong]{} \pi_k(X')$ for $k < n$
  2. surjective for $k = n$

- Given X, there is $i : X \to Y = X \cup (n+1)$-cells such that

  1. $\pi_k(X) \xrightarrow{i_\#} \pi_k(Y)$ is isomorphism for $k < n$
  2. $\pi_n(Y) = 0$

By using this we can construct spaces

$$\begin{aligned} Y_1 &= X \cup e^3 \cup \cdots = K(\pi_1, 1) \\ &\vdots \\ Y_n &= X \cup e^{n+2} \ldots \quad \text{such that } \pi_i(Y_n) = \begin{cases} 0 & i \geq n+1 \\ \pi_i & i = 1, 2, \ldots, n \end{cases} \end{aligned}$$

Here, Y_n is obtained by killing off homotopy of X in dim $\geq n+1$.

Constructing Y_{n-1} from Y_n: We can obtain Y_{n-1} by killing the homotopy of Y_n in $dim \geq n$, since $Y_{n-1} = Y_n \cup e^{n+1} \cup \ldots$. we have inclusions, $X \subset Y_n \subset Y_{n-1} \subset \cdots \subset Y_1$. Postnikov tower is a process of converting these inclusions to fibrations. For this we need to review some basic facts about fibrations:

Remark 7.1. (1) Given a fibration $p : E \to B$ and a map $f : X \to B$ consider the pullback fibration (X, E') and a map $g : Y \to X$. Then g has a lifting $h : Y \to E'$ if $f \circ g \simeq *$.

$$\begin{tikzcd}
 & E' \arrow[r, "\pi_2"] \arrow[d] & E \arrow[d, "p"] \\
Y \arrow[ur, dashed, bend left, "h"] \arrow[urr, dashed, "h_E"] \arrow[r, "g"'] & X \arrow[r, "f"'] & B
\end{tikzcd}$$

To see this, write $E' = \{(x, e) \mid f(x) = p(e)\} \subset X \times E$, then use HLP to lift $f \circ g$ to $h_E : Y \to E$ with $p \circ h_E = f \circ g$. Then let $h = g \times h_E : Y \to E' \subset X \times E$.

(2) Let $\xi = (E, p, B, F)$ be a fibration and F $(n-1)$-connected. Then exists $i_F \in \mathrm{H}^n(F; \pi_n F)$, which transgresses: $\tau(i_F) \in \mathrm{H}^n(B; \pi_n F)$ ()called characteristic class).

(3) Consider the path space fibration $K(\pi, n) \to E \to K(\pi, n+1)$, where $E \simeq *$

$$\begin{array}{ccc}
\tau : \mathrm{H}^n(K(\pi, n); \pi) & \longrightarrow & \mathrm{H}^{n+1}(K(\pi, n+1); \pi) \\
\Cup & & \Cup \\
i_n & \dashrightarrow & i_{n+1}
\end{array}$$

(4) Consider the induced (pull-back) fibration by $f : X \to K(\pi, n+1)$.

$$\begin{tikzcd}
K(\pi, n) \arrow[r, "\pi_2"] \arrow[d] & K(\pi, n) \arrow[d] \\
E' \arrow[r] \arrow[d, "q"'] & E \simeq * \arrow[d, "p"'] \\
X \arrow[r, "f"'] & K(\pi, n+1)
\end{tikzcd}$$

Then the induced Serre exact sequences imply $f^*(i_{n+1}) = \tau_q(i_n)$ (when defined)

$$\begin{tikzcd}
\arrow[r] & \mathrm{H}^n(K(\pi, n)) \arrow[r, "\tau_p"] \arrow[d, "="] & \mathrm{H}^{n+1}(K(\pi, n+1)) \arrow[r, "p^*"] \arrow[d, "f^*"] & \mathrm{H}^{n+1}(E) \arrow[d, "\pi_2^*"] \\
\arrow[r] & \mathrm{H}^n(K(\pi, n)) \arrow[r, "\tau_q"] & \mathrm{H}^{n+1}(X) \arrow[r, "q^*"] & \mathrm{H}^{n+1}(E')
\end{tikzcd}$$

7.1 Postnikov Tower

Let X be $(n-1)$-connected $(n \geq 2)$, call $\pi_i = \pi_i(X)$, we construct a tower, such that

$$\begin{tikzcd}
 & & {} \arrow[d] & \\
 & & X_{m+1} \arrow[d] & \\
 & & X_m \arrow[r, "k_m"] \arrow[d] & K(\pi_{m+1}, m+2) \\
 & & \vdots \arrow[d] & \begin{pmatrix} \textit{homotopy} \\ \textit{commutative} \end{pmatrix} \\
 & & X_{n+1} \arrow[d] & \\
F_n \arrow[r] & X \arrow[r, "f = \rho_n"'] \arrow[uuuur, bend left, "\rho_{m+1}"] \arrow[uuur, bend left, "\rho_m"] \arrow[ur, bend left, "\rho_{n+1}"] & X_n = K(\pi_n, n) \arrow[r, dashed, "k_n"] & K(\pi_{n+1}, n+2)
\end{tikzcd}$$

- $\rho_m \simeq$ on π_i for $i \leq m$
- $\pi_i(X_m) = 0$ for $i > m$
- X_{m+1} is induced by some $k_m \in H^{m+2}(X_m, \pi_{n+1})$ (called the haracteristic element)

Existance: Start by $f : X \to K(\pi_n, n) := X_n$, which is isomorphism on π_i for all $i \leq n$. Then make f a fibration, and let F_n be the homotopy theoretical fiber of f. Then by the homotopy exact sequence of fibration $F_n \xrightarrow{j_n} X \xrightarrow{f} X_n$ with fiber F_n satisfying $\pi_i(F_n) = \pi_i$ for $i \geq n+1$ and $\pi_i(F_n) = 0$ for $i \leq n$. Then pick the identity element

$$i_n \in \mathrm{H}^{n+1}(F_n, \pi_{n+1}) \cong \mathrm{Hom}(\mathrm{H}_{n+1}(F_n), \pi_{n+1}) \cong \mathrm{Hom}(\pi_{n+1}, \pi_{n+1})$$

and define $k_n = \tau(i_n) \in H^{n+2}(X_n, \pi_{n+1})$, by the Serre exact sequence (Theorem 6.12)

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^{n+1}(F_n;\pi_{n+1}) \arrow[r, "\tau"] & \mathrm{H}^{n+2}(X_n;\pi_{n+1}) \arrow[r, "f^*"] & \mathrm{H}^{n+2}(X,\pi_{n+1}) \arrow[r] & \dots \\
& i_n \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, dashed] & k_n \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, dashed] & 0 &
\end{tikzcd}$$

so $f^*(k_n) = 0$. By considering the homology class as a map $k_n : X_n \to K(\pi_{n+1}, n+2)$ we get a null homotopy $k_n \circ f \simeq *$. By Remark 7.1 (1) this implies that f lifts to the total space of the fibration $X_{n+1} \to X_n$ induced by the map k_n, from the path space (universal) fibration $K(\pi_{n+1}, n+1) \to * \to K(\pi_{n+1}, n+2)$. Call this lifting $g : X \to X_{n+1}$. Usually this total space X_{n+1} is denoted by $X_{n+1} = K(\pi_n, n) \times_{k_n} K(\pi_{n+1}, n+1)$.

$$\begin{tikzcd}
& & & \text{induced fibration} \arrow[dl] & \\
K = & K(\pi_{n+1}, n+1) \arrow[r] & X_{n+1} \arrow[d, "p"] \arrow[rr, dashed] & & * \arrow[d] \\
& F_n \arrow[u, dashed, "s"] \arrow[r, "j_n"'] & X \arrow[u, dashed, "g"] \arrow[r, "f"'] & X_n \arrow[r, "k_n"'] & K(\pi_{n+1}, n+2)
\end{tikzcd}$$

We claim that g can be chosen so that $g_{\#} : \pi_i(F_n) \xrightarrow{\approx} \pi_i(X_{n+1})$ for $i \leq n+1$. Note that the claim implies the existence of Postnikov tower (by iterating this process). $\square$

Proof of Claim:

- Notice that since $g(f^{-1}(x)) \subset p^{-1}(x)$, the map g maps fibers to fibers $F_n \to K(\pi_{n+1}, n+1)$. Call this restriction map $s : F_n \to K(\pi_{n+1}, n+1)$, which can be considered $s \in \mathrm{H}^{n+1}(F_n, \pi_{n+1})$. For brevity call $K = K(\pi_{n+1}, n+1)$.

- By construction fundamental class $i_{n+1} \in \mathrm{H}^{n+1}(K)$ transgress to k_n, by naturality $s \in \mathrm{H}^{n+1}(F_n)$ transgress k_n; and also by definition $i_{n+1} \in \mathrm{H}^{n+1}(F_n)$ transgress k_n.

- Since $\tau(i_{n+1}) = \tau(s)$, $s = i_{n+1} + j_n^*(\alpha)$ for some $\alpha \in \mathrm{H}^{n+1}(X, \pi_{n+1})$ (Serre sequence).

$$\begin{tikzcd}
 & \text{induced fiber space} \arrow[d] & \\
 & K = K(\pi_{n+1}, n+1) \arrow[r, equal] \arrow[d] & K(\pi_{n+1}, n+1) \arrow[d] \\
F_n \arrow[ur, dashed, "s"] \arrow[d, "j_n"] & X_{n+1} \arrow[r, dashed] \arrow[d, "p"'] & * \arrow[d] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "g"] & K(\pi_n, n) = X_n \arrow[r, dashed, "k_n"'] & K(\pi_{n+1}, n+2)
\end{tikzcd}$$

The pull-back fibration $K \to X_{n+1} \to X_n$ is a principal fiber space with the action $\varphi : X_{n+1} \times K \to X_{n+1}$ of $K = \Omega K(\pi_{n+1}, n+2)$, by loop multiplication, which corresponds to addition in $\mathrm{H}^{n+1}(F_n, \pi_{n+1})$. Now we can redefine g as g' by

$$\begin{tikzcd}
g' : X \arrow[r, "(g, -\alpha)"'] \arrow[rr, dashed, bend left, "g'"] & X_{n+1} \times K \arrow[r, "\varphi"'] & X_{n+1}
\end{tikzcd}$$

$\alpha : X \to K$ since $\alpha \in \mathrm{H}^{n+1}(X, \pi_{n+1})$. Thus $pg = pg' = f$, and recall $s = g|_{F_n}$. So $s = \varphi \circ (s, -j_n^*\alpha) = s - j_n^*(\alpha) = i_{n+1}$. This implies that the map g' is $\cong$ on π_{n+1}. $\square$

Remark 7.2. Postnikov decomposition is usually called the atomic decomposition in homotopy theory, where $K(\pi, n)$'s corresponds atoms, which are glued together by k_n's to make molecules. That is, every $(n-1)$-connected space X has the decomposition.

$$X \simeq K(\pi_n, n) \times_{k_n} K(\pi_{n+1}, n+1) \times_{k_{n+1}} K(\pi_{n+1}, n+1)..$$

7.2 Whitehead Tower

Let X be a connected space. The Whitehead tower is the following fibrations, such that

$$\begin{tikzcd} X_n \arrow[d] & K(\pi_n, n-1) \arrow[l] \\ \vdots \arrow[d] & \\ X_2 \arrow[d] & K(\pi_2, 1) \arrow[l] \\ X_1 \arrow[d] & K(\pi_1, 0) \arrow[l] \\ X & \end{tikzcd}$$

(i) $\pi_k(X_n) = 0$ for $k \leq n$

(ii) $\pi_k(X_n) \xrightarrow{\cong} \pi_k(X)$ for $k \geq n+1$

(iii) $\begin{tikzcd} K(\pi_n, n-1) \arrow[r] & X_n \arrow[d] \\ & X_{n-1} \end{tikzcd}$ is a fibration.

Constructing X_n from X_{n-1}:

Proof. First construct $K(\pi_n, n)$ by attaching cells to X_{n-1}

$$K(\pi_n, n) = X_{n-1} \cup e^{n+2} \cup \dots$$

$$X_n = \{\sigma : I \to K(\pi_n, n) \mid \sigma(0) = *,\ \sigma(1) \in X_{n-1}\}$$

The map $\sigma \mapsto \sigma(1)$ gives a fibration $X_n \to X_{n-1}$ fitting into pullback diagram:

$$\begin{tikzcd} \Omega K(\pi_n, n) \arrow[r, "="'] \arrow[d] & K(\pi_n, n-1) \arrow[d] \\ X_n \arrow[r, dashed] \arrow[d] & * \arrow[d] \\ X_{n-1} \arrow[r, hook] & K(\pi_n, n) \end{tikzcd} \tag{7.1}$$

Homotopy exact sequence of fibration implies $\pi_n(X_n) = \pi_{n-1}(X_n) = 0$

$$0 \longrightarrow \pi_n(X_n) \longrightarrow \pi_n(X_{n-1}) \xrightarrow{\partial} \pi_{n-1}(\Omega K(\pi_n, n)) \longrightarrow \pi_{n-1}(X_n) \longrightarrow 0$$

$$\begin{array}{ccc} \text{induction} \longrightarrow \wr\| & & \wr\| \\ \pi_n & & \pi_n \end{array}$$

$$\begin{tikzcd} 0 \arrow[r] & \pi_n(K(\pi_n, n)) \arrow[r, "\partial", "\approx"'] & \pi_{n-1}(\Omega K(\pi, n)) \arrow[r] \arrow[d, equal] & 0 \\ & \pi_n(X_{n-1}) \arrow[u, "\approx"] \arrow[r, "\partial"] & \pi_{n-1}(\Omega K(\pi, n)) \arrow[r] & 0 \end{tikzcd}$$

$$\text{Therefore} \quad \pi_i(X_n) \cong \begin{cases} \pi_i(X_{n-1}) & \text{if } i \geq n+1 \\ 0 & \text{if } i \leq n \end{cases}$$

□

Example 7.1. $\pi_4(S^3) = \mathbb{Z}_2$ *and* $\pi_5(S^3) = \mathbb{Z}_2$*. Consider Whitehead tower.*

$$\begin{tikzcd} K(\pi_4, 3) \arrow[r] & X_4 \arrow[d] & \arrow[l, dashed] 4\text{-connected} \\ K(\mathbb{Z}, 2) = K(\pi_3, 2) \arrow[r] & X_3 \arrow[d] & \arrow[l, dashed] 3\text{-connected} \\ & S^3 & \end{tikzcd}$$

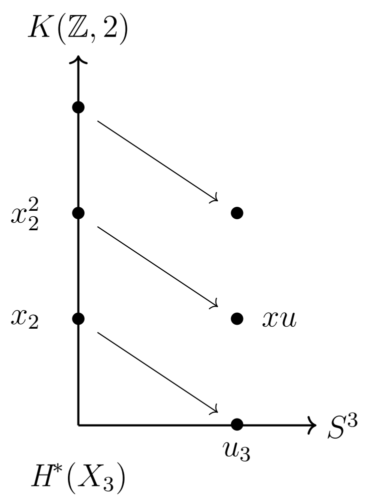
By using homotopy exact sequence fibrations and Hurewicz theorem we calculate:

$$\pi_4(S^3) \cong \pi_4(X_3) \cong H_4(X_3) \cong \mathbb{Z}_2$$
$$\pi_5(S^3) \cong \pi_5(X_3) \cong \pi_5(X_4) \cong H_5(X_4) \cong \mathbb{Z}_2$$

We calculate the last isomorphisms by the spectral sequence of corresponding fibrations.

$$d_2 : E_2^{0,2n} \to E_2^{2,2n+1}$$

$d_2(x_2^n) = nx_2^{n-1}d_2(x_2) = nx_2^{n-1}u_3$

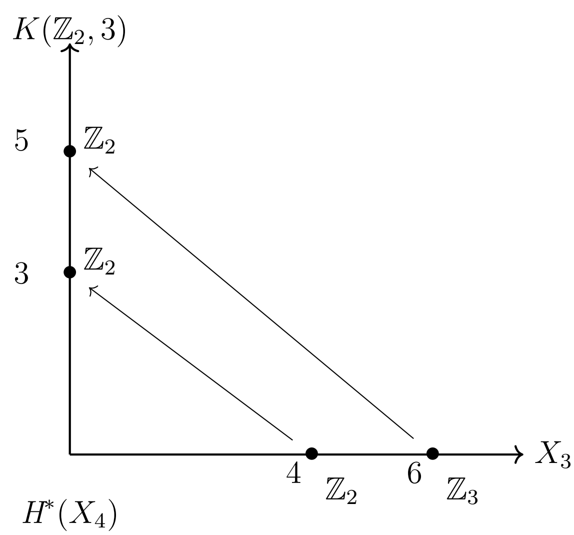


	0	1	2	3	4	5	6	7	8	9	10	11
$H^q(X_3)$	$\mathbb{Z}$					$\mathbb{Z}_2$		$\mathbb{Z}_3$		$\mathbb{Z}_4$		$\mathbb{Z}_5$
$H_q(X_3)$	$\mathbb{Z}$				$\mathbb{Z}_2$		$\mathbb{Z}_3$		$\mathbb{Z}_4$		$\mathbb{Z}_5$	

Hence $\pi_4(S^3) = \mathbb{Z}_2$. *To calculate* $H_5(X_4)$ *we we can use the homology spectral sequence of the fibration* $K(\mathbb{Z}_2, 3) \to X_4 \to X_3$. *Since* X_4 *is* 4*-connected.*

$$d_4 : E^4_{4,0} = \mathbb{Z}_2 \xrightarrow{\cong} \mathbb{Z}_2 = E^4_{0,3} \quad \text{and} \quad d_6 : E^6_{6,0} = \mathbb{Z}_3 \xrightarrow{0} \mathbb{Z}_2 = E^6_{0,5}$$

Hence $H_5(X_4) \cong Z_2$, *and so* $\pi_5(S^3) \cong \mathbb{Z}_2$. *Here we used* $H^5(K(\mathbb{Z}_3, 3)) = \mathbb{Z}_2$.



$$\left\{ \begin{array}{c} H_4(K(\mathbb{Z}_2,3)) = 0 \\ H_5(K(\mathbb{Z}_2,3)) = \mathbb{Z}_2 \end{array} \right\}$$

From the spectral sequence of fibrations $RP^\infty = K(\mathbb{Z}_2, 1) \to * \to K(\mathbb{Z}_2, 2)$ *and* $K(\mathbb{Z}_2, 2) \to * \to K(\mathbb{Z}_2, 3)$ *we can easily compute* $H_3(K(\mathbb{Z}_2, 3)) \cong H_3(K(\mathbb{Z}_2, 5)) \cong \mathbb{Z}_2$. *Alternatively, we can also compute* $\pi_4(S^3) = \mathbb{Z}_2$ *by using the Postnikov tower of* S^3.

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